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Decomposition of Spectra from the Drum

With Applications to the Chesapeake Bay

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Outline

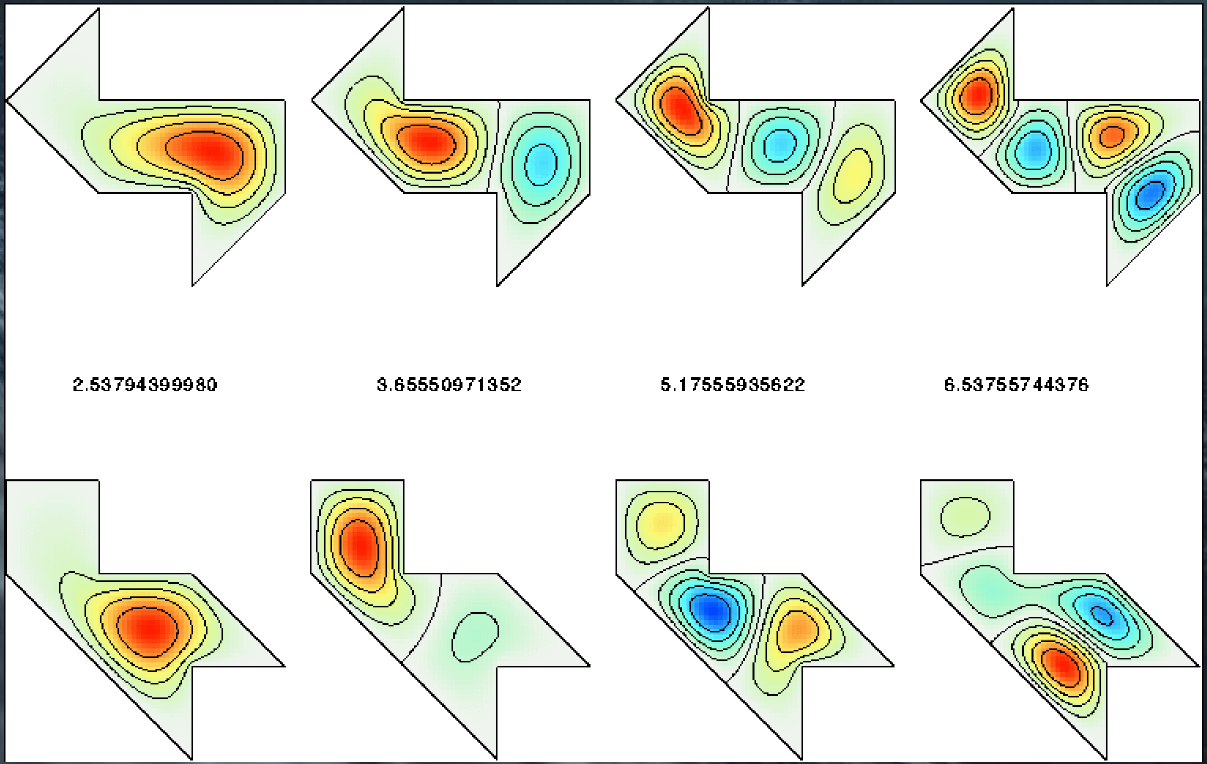
- Review of Problem Statement / History
- Motivation / Applications
- Methods Applied – Toy Problems
- Analysis
- Conclusions

Drumhead Problem

- Given a sample of a drums sound, attempt to calculate the amplitudes of the modes
  - time-series at location of microphone
- Related to a famous problem posed by mathematician Mark Kac (1966) asking: "Can One Hear the Shape of a Drum?"
- Lead to idea of iso-spectral drums

Isospectral Drums

- Drums with differing boundaries that have identical k_n values – so they sound alike!



Milnor, 1966 …
Driscoll, 1997 SIAM

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Why Measure the Amplitudes?

- Being able to measure modes strengths suggests dynamics about the system.
- By measuring “windows” in time that overlap, the time-dependence of the amplitudes can be seen.
- Energy conservation – once a mode is excited, where does the energy go?
- Lead to prediction of amplitudes beyond the time window (forecasting).

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Methodology

- Three key issues:
  - Delta Function
  - Solution Architecture
  - Degeneracy of States
- Midn 1/C Grant Hundley
  - Developed a drum simulator
  - Working on numerical scheme to extract amplitudes from simulated time-series

Delta Function

$$
\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty}\left[a_n(t)\sin(k_n x) + b_n(t)\cos(k_n x)\right] \\
a_m(t) &= \frac{1}{L}\int_{-L}^{+L} u(x,t)_{data}\sin(k_m x)dx \\
a_m(t) &= \frac{1}{L}\int_{-L}^{+L}\sum_{n=0}^{\infty}\left[a_n(t)\sin(k_n x) \;+\; b_n(t)\cos(k_n x)\right]\sin(k_m x)dx \\
a_m(t) &= \sum_{n=0}^{\infty}\left[a_n(t)\frac{1}{L}\int_{-L}^{+L}\sin(k_n x)\sin(k_m x)dx \;+\; b_n(t)\frac{1}{L}\int_{-L}^{+L}\cos(k_n x)\sin(k_m x)dx\right] \\
a_m(t) &= \sum_{n=0}^{\infty}\left[a_n(t)\delta_{nm} + b_n(t)\o\right] \\
a_m(t) &= \Delta_{mn}a_n(t) \\
a_n(t) &= \oint u(x,t)_{data}\sin(k_n x)dx \\
b_n(t) &= \oint u(x,t)_{data}\cos(k_n x)dx
\end{aligned}
$$

Delta Function Assumptions

$$\omega_n = \frac{n*\pi}{2T} \text{ and } \omega_m = \frac{m*\pi}{2T}$$

ω_n and ω_m have integer relationship

$$\delta_{mn} = \frac{1}{T}\int_{-T}^{+T} \sin(\omega_n t)\sin(\omega_m t)dt$$

$$(T_1 = 4T)$$

Wavelength is set from the window and is symmetric

$$\emptyset = \frac{1}{T}\int_{-T}^{+T} \cos(\omega_n t)\sin(\omega_m t)dt$$

$$\Delta_{mn} = \frac{1}{T}\int_{-T}^{+T} \sin(\omega_n t)\sin(\omega_m t)dt$$

$$\varepsilon_{mn} = \frac{1}{T}\int_{-T}^{+T} \cos(\omega_n t)\sin(\omega_m t)dt$$

Define: Delta and Epsilon

Should these conditions fail.

Delta Function

$$\Delta_{mn} = \frac{1}{T}\int_{-T}^{+T} \sin(\omega_n t) \quad \sin(\omega_m t) dt$$

$$\Delta_{mn} = \frac{1}{2T}\int_{-T}^{+T} \left[\cos\left((\omega_n - \omega_m)t\right) \quad - \quad \cos\left((\omega_n + \omega_m)t\right)\right] dt$$

$$\Delta_{mn} = \frac{1}{2T}\frac{1}{\omega_n - \omega_m}\sin\left((\omega_n - \omega_m)x\right)\Big|_{-T}^{+T} \quad - \quad \frac{1}{2T}\frac{1}{\omega_n + \omega_m}\sin\left((\omega_n + \omega_m)x\right)\Big|_{-T}^{+T}$$

$$\Delta_{mn} = \frac{1}{2T}\frac{1}{\frac{(n-m)\pi}{2T}}\sin\left(\frac{(n-m)\pi}{2T}x\right)\Big|_{-T}^{+T} \quad - \quad \frac{1}{2T}\frac{1}{\frac{(n+m)\pi}{2T}}\sin\left(\frac{(n+m)\pi}{2T}x\right)\Big|_{-T}^{+T}$$

$$\Delta_{mn} = \frac{2}{\pi(n-m)}\sin(\frac{\pi}{2}(n-m)) \quad - \quad \frac{2}{\pi(n+m)}\sin(\frac{\pi}{2}(n+m)), \quad \text{for even } n \rightarrow 2n$$

$$\Delta_{mn} = \text{sinc}(n-m) \quad - \quad \text{sinc}(n+m), \qquad \text{assuming } m \text{ is free (real)}$$

$$\Delta_{\pm m, 2n} = \delta_{\pm m, 2n}, \qquad \text{assuming } n \text{ is even amd } m \text{ is an integer}$$

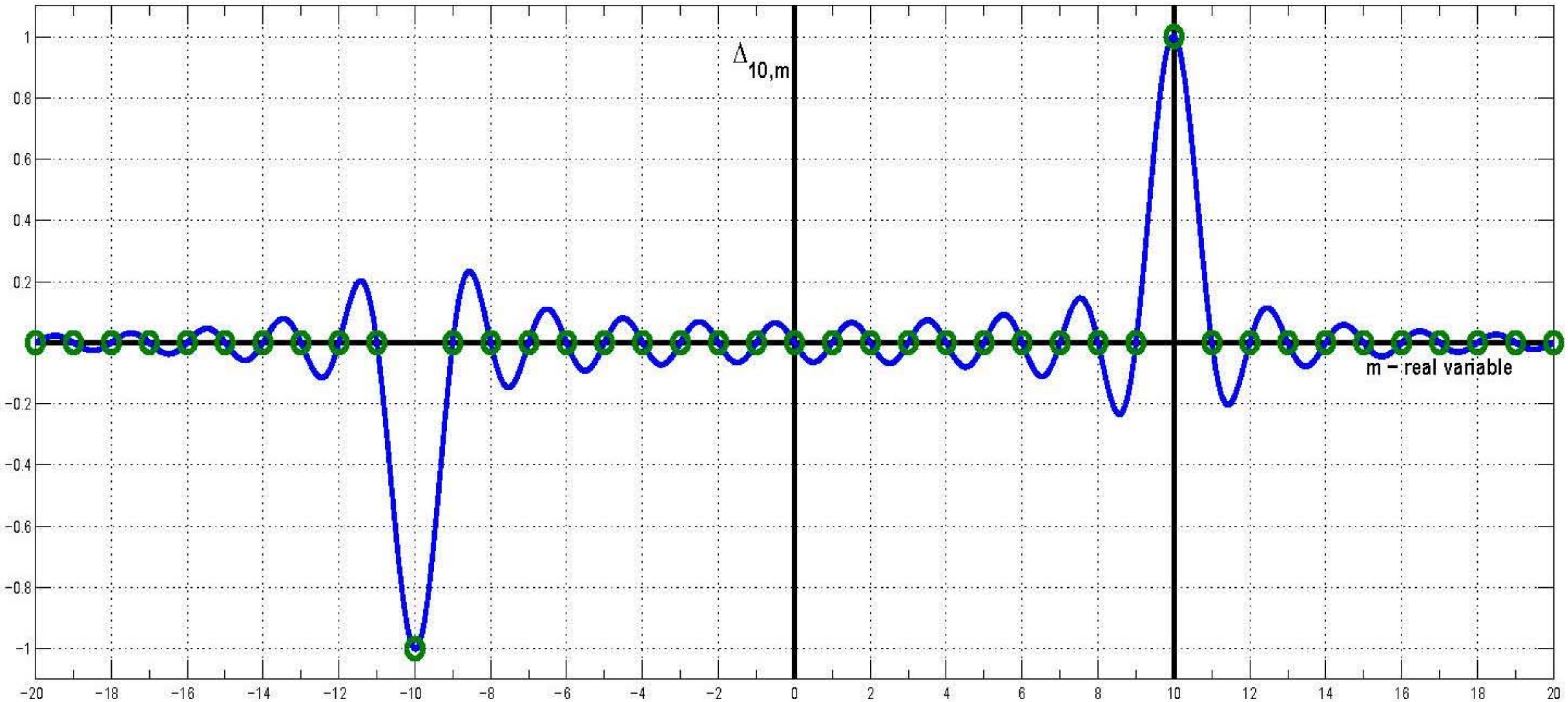
$$\varepsilon_{mn} = \frac{1}{T}\int_{-T}^{+T} \sin(\omega_n t) \quad \cos(\omega_m t) dt$$

$$\varepsilon_{mn} = \frac{1}{2T}\int_{-T}^{+T} \left[\sin\left((\omega_n - \omega_m)t\right) - \sin\left((\omega_n + \omega_m)t\right)\right] dt$$

$$\varepsilon_{mn} = \emptyset$$

Delta Function ~ Sinc(pi*(n-m))

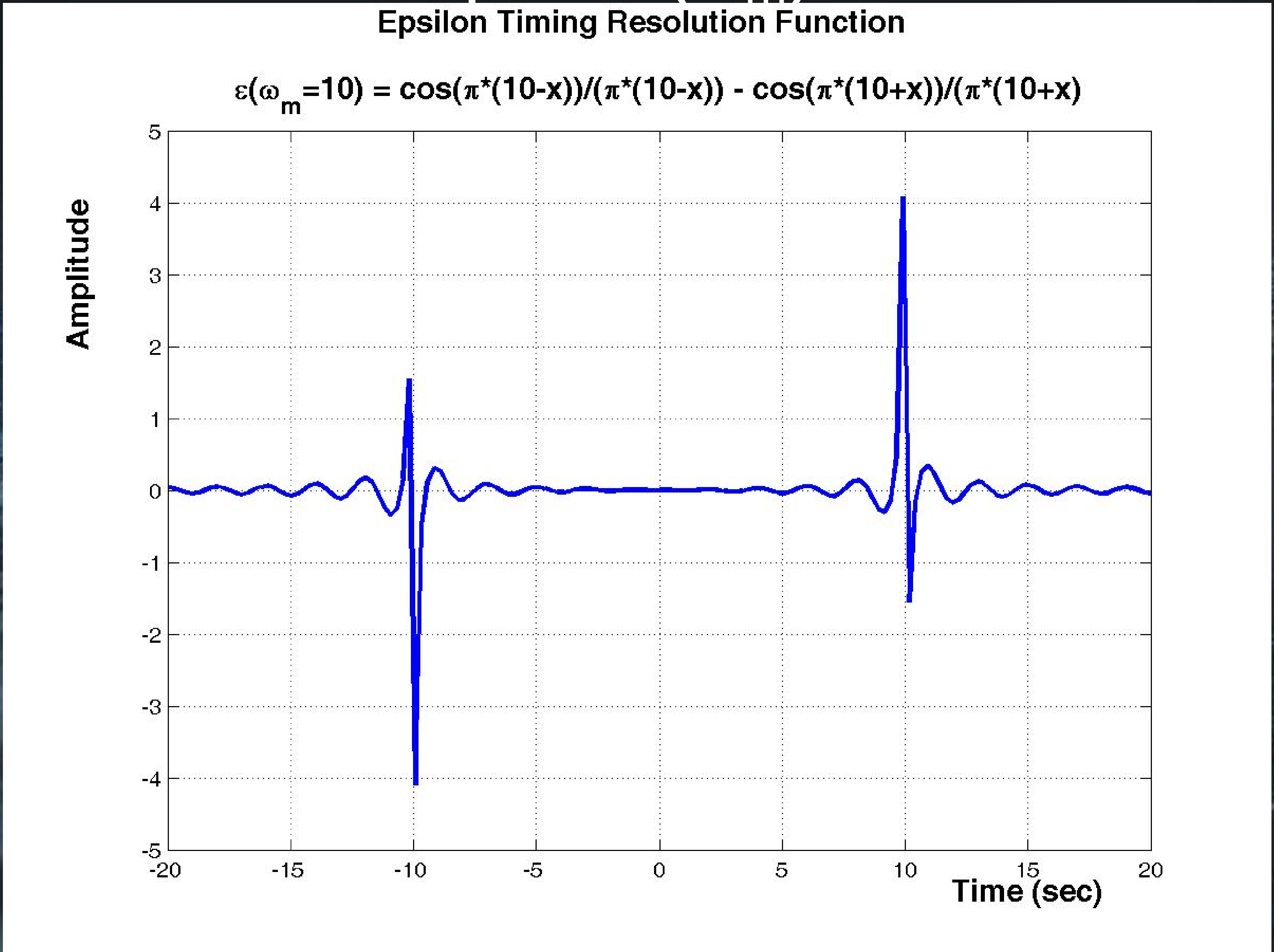
The Δ_{nm} function shown below shows its clear approximation to δ_{nm} when (n, m) are integers. Also shown are the twin responses at $(-m, +m)$ for the $n = 10$ case.



Delta & Epsilon Function

- Delta function is a measure of the “mixing” between the eigenmodes.
- Slightly different Delta fucntion for the cos*cos term (more on this later).
- Epsilon will be identically zero for symmetric domains.

Epsilon(ω_m)



Solution Architecture:
Solutions to space-time problems

$$u(x,t) = f(x) \cdot g(t)$$

$$f(x) = \sum_{n=0}^{\infty} A_n f_D(k_n x) + B_n f_N(k_n x)$$

$$g(t) = \sum_{n'=0}^{\infty} C_{n'} g_D(\omega_{n'} t) + D_{n'} g_N(\omega_{n'} t)$$

$$u(x,t) = \left[\sum_{n=0}^{\infty} A_n f_D(k_n x) + B_n f_N(k_n x)\right]\left[\sum_{n'=0}^{\infty} C_{n'} g_D(\omega_{n'} t) + D_{n'} g_N(\omega_{n'} t)\right]$$

$$u(x,t) = \left[\sum_{n=0}^{\infty} AC_n f_{D,n} g_{D,n} + BC_n f_{N,n} g_{D,n} + AD_n f_{D,n} g_{N,n} + BD_n f_{N,n} g_{N,n}\right]$$

$$u(x,t) = \left[\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} AC_{nn'} f_{D,n} g_{D,n'} + BC_{nn'} f_{N,n} g_{D,n'} + AD_{nn'} f_{D,n} g_{N,n'} + BD_{nn'} f_{N,n} g_{N,n'}\right]$$

Nature of Eigenmodes with both space and time in solution

- Two types:
  - Coupled - for each k eigenvalue in space there exists a unique ω eigenvalue in time
    - Dispersion relationship, $\omega(k)$, for most differential equations in (x,t).
    - Dispersion relationship is generally monotonic and increasing.
  - Decoupled – for systems not well motivated physically, yet can be described with a space-time basis set (river bank problem).

Solution Architecture

$$\alpha_m = \frac{1}{T}\int_{-T}^{+T} \sin(\omega_m t)\ \ u(x_1, t)\ \ dt$$

$$\beta_m = \frac{1}{T}\int_{-T}^{+T} \cos(\omega_m t)\ \ u(x_1, t)\ \ dt$$

At one location, x1, sample the data in a time-series.

Project out the sin and cosines.

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \\ \hline \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{pmatrix} = \underbrace{\frac{1}{T}\int_{-T}^{+T} \begin{pmatrix} g_D(\omega_1 t) \\ g_D(\omega_2 t) \\ g_D(\omega_3 t) \\ \vdots \\ g_D(\omega_m t) \\ \hline g_N(\omega_1 t) \\ g_N(\omega_2 t) \\ g_N(\omega_3 t) \\ \vdots \\ g_N(\omega_m t) \end{pmatrix} \otimes \left(g_D(\omega_1 t)\ g_D(\omega_2 t)\ \ldots\ g_D(\omega_n t) \,\middle|\, g_N(\omega_1 t)\ g_N(\omega_2 t)\ \ldots\ g_N(\omega_n t) \right) dt}_{\text{outer product becomes matrix of } \Delta_{mn} \text{ and } \varepsilon_{mn}} \left(\begin{array}{ccccc|ccccc} f_D(k_1 x_1) & & & & & & & & & \\ & f_D(k_2 x_1) & & & & & & & & \\ & & f_D(k_3 x_1) & & & & & 0 & & \\ & & & \ddots & & & & & & \\ & & & & f_D(k_n x_1) & & & & & \\ \hline & & & & & f_D(k_1 x_1) & & & & \\ & & & & & & f_D(k_2 x_1) & & & \\ & & 0 & & & & & f_D(k_3 x_1) & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & f_D(k_n x_1) \end{array} \right) \begin{pmatrix} AC_1 \\ AC_2 \\ AC_3 \\ \vdots \\ AC_n \\ \hline AD_1 \\ AD_2 \\ AD_3 \\ \vdots \\ AD_n \end{pmatrix}$$

Solution Architecture

$$
\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \\ \hline \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{pmatrix}
=
\left(\begin{array}{ccccc|ccccc}
\Delta_{11} & \Delta_{12} & \Delta_{13} & \cdots & \Delta_{1n} & \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \cdots & \varepsilon_{1n} \\
\Delta_{21} & \Delta_{22} & \Delta_{23} & & \Delta_{2n} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & & \varepsilon_{2n} \\
\Delta_{31} & \Delta_{32} & \Delta_{33} & & \Delta_{3n} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} & & \varepsilon_{3n} \\
\vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\
\Delta_{m1} & \Delta_{m2} & \Delta_{m3} & \cdots & \Delta_{mn} & \varepsilon_{m1} & \varepsilon_{m2} & \varepsilon_{m3} & \cdots & \varepsilon_{mn} \\
\hline
\varepsilon_{11} & \varepsilon_{21} & \varepsilon_{31} & \cdots & \varepsilon_{n1} & \Delta^{\circ}_{11} & \Delta^{\circ}_{12} & \Delta^{\circ}_{13} & \cdots & \Delta^{\circ}_{1n} \\
\varepsilon_{12} & \varepsilon_{22} & \varepsilon_{32} & & \varepsilon_{n2} & \Delta^{\circ}_{21} & \Delta^{\circ}_{22} & \Delta^{\circ}_{23} & & \Delta^{\circ}_{2n} \\
\varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} & & \varepsilon_{n3} & \Delta^{\circ}_{21} & \Delta^{\circ}_{32} & \Delta^{\circ}_{33} & & \Delta^{\circ}_{3n} \\
\vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\
\varepsilon_{1m} & \varepsilon_{2m} & \varepsilon_{3m} & \cdots & \varepsilon_{nm} & \Delta^{\circ}_{m1} & \Delta^{\circ}_{m2} & \Delta^{\circ}_{m3} & \cdots & \Delta^{\circ}_{mn}
\end{array}\right)
\left(\begin{array}{c|c}
\begin{matrix} f_D(k_1 x_1) & & & \\ & f_D(k_2 x_1) & & \\ & & f_D(k_3 x_1) & \\ & & & \ddots \\ & & & & f_D(k_n x_1) \end{matrix} & 0 \\
\hline
0 & \begin{matrix} f_D(k_1 x_1) & & & \\ & f_D(k_2 x_1) & & \\ & & f_D(k_3 x_1) & \\ & & & \ddots \\ & & & & f_D(k_n x_1) \end{matrix}
\end{array}\right)
\begin{pmatrix} AC_1 \\ AC_2 \\ AC_3 \\ \vdots \\ AC_n \\ \hline AD_1 \\ AD_2 \\ AD_3 \\ \vdots \\ AD_n \end{pmatrix}
$$

- Mixing nature of Delta,Epsilon matrix can clearly be seen.
- No nodes can exist in the f(k*x) matrix.
- Solve for Amplitudes thru matrix inversion.

Solution Architecture

- Short-hand notation:

$$\begin{pmatrix} \overrightarrow{\alpha} \\ \overrightarrow{\beta} \end{pmatrix} = \begin{pmatrix} \Delta_t & \varepsilon_t \\ \varepsilon_t^{\dagger} & \Delta_t^{c} \end{pmatrix} \begin{pmatrix} \overrightarrow{f_D} & \text{O} \\ \text{O} & \overrightarrow{f_D} \end{pmatrix} \begin{pmatrix} \overrightarrow{AC} \\ \overrightarrow{AD} \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \Delta_t & \varepsilon_t \\ \varepsilon_t^{\dagger} & \Delta_t^{c} \end{pmatrix} \begin{pmatrix} f_D & \text{O} \\ \text{O} & f_D \end{pmatrix} \begin{pmatrix} AC \\ AD \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{T} \int_{-T}^{+T} \begin{pmatrix} g_D \\ g_N \end{pmatrix} \otimes^{(g_D \; g_N)} \begin{pmatrix} f_D & \text{O} \\ \text{O} & f_D \end{pmatrix} \begin{pmatrix} AC \\ AD \end{pmatrix} dt = \frac{1}{T} \int_{-T}^{+T} \begin{pmatrix} g_D \\ g_N \end{pmatrix} \otimes^{(g_D \; g_N)} \begin{pmatrix} AC \cdot f_D \\ AD \cdot f_D \end{pmatrix} \; dt$$

$$\begin{pmatrix} \alpha_m \\ \beta_m \end{pmatrix} = \frac{1}{T} \int_{-T}^{+T} \begin{pmatrix} g_{Dm} \\ g_{Nm} \end{pmatrix} \otimes \sum_{n=0}^{\infty} \left[AC_n \cdot f_D(k_n \cdot x_1) \cdot g_{Dn} + AD_n \cdot f_D(k_n \cdot x_1) \cdot g_{Nn} \right] dt = \frac{1}{T} \int_{-T}^{+T} \begin{pmatrix} g_{Dm} \\ g_{Nm} \end{pmatrix} \otimes u(x_1, t) \; dt$$

Drum – Specific Solutions

- Drum problem: membrane stretched over a circular boundary (Dirichlet bc).
- Strike the drum.
- Use a microphone to record the time-series.
- Fourier analyze the time-series to obtain amplitudes for each ω_m : A_n

Drum

$$u(x,t)=\sum_{n=0}^{\infty}\left[A_n J_n(k_n r)\sin(n\theta)+B_n J_n(k_n r)\cos(n\theta)\right]\left[C_n\sin(\omega_n t)+D_n\cos(\omega_n t)\right]$$

$$u(r,t)=\sum_{(n,n')=0}^{\infty}\left[A_{nn'} J_n(k_{nn'} r)\sin(n\theta)+B_{nn'} J_n(k_{nn'} r)\cos(n\theta)\right]\left[C_{nn'}\sin(\omega_{nn'} t)+D_{nn'}\cos(\omega_{nn'} t)\right]$$

- Spatial modes are a combination of Bessel function, $J(k_n r)$ times $\sin(n\theta)$ or $\cos(n\theta)$
- Temporal modes use $\sin(\omega_n t)$ or $\cos(\omega_n t)$
- For each Bessel function, there exists multiple zero crossings, n'
- k_n values are non-integerlike, so ω_n fail conditions for orthonormality

$$\begin{pmatrix} f_{D1} \\ f_{D2} \\ f_{D3} \\ f_{D4} \\ f_{D5} \\ f_{D6} \\ f_{D7} \\ f_{D8} \\ f_{D9} \\ f_{D10} \\ f_{D11} \\ f_{D12} \\ f_{D13} \\ f_{D14} \\ f_{D15} \\ f_{D16} \\ f_{D17} \\ f_{D18} \\ f_{D19} \\ f_{D20} \\ \vdots \\ f_{Dn_{max}} \\ \hline f_{D1} \\ f_{D2} \\ f_{D3} \\ \vdots \\ f_{Dn_{max}} \end{pmatrix} = \begin{pmatrix} J_0(k_{01}r_1)\sin(0\ \theta_1) \\ J_0(k_{01}r_1)\cos(0\ \theta_1) \\ J_1(k_{11}r_1)\sin(1\ \theta_1) \\ J_1(k_{11}r_1)\cos(1\ \theta_1) \\ J_2(k_{21}r_1)\sin(2\ \theta_1) \\ J_2(k_{21}r_1)\cos(2\ \theta_1) \\ J_0(k_{02}r_1)\sin(0\ \theta_1) \\ J_0(k_{02}r_1)\cos(0\ \theta_1) \\ J_3(k_{31}r_1)\sin(3\ \theta_1) \\ J_3(k_{31}r_1)\cos(3\ \theta_1) \\ J_1(k_{12}r_1)\sin(1\ \theta_1) \\ J_1(k_{12}r_1)\cos(1\ \theta_1) \\ J_4(k_{41}r_1)\sin(4\ \theta_1) \\ J_4(k_{41}r_1)\cos(4\ \theta_1) \\ J_2(k_{22}r_1)\sin(2\ \theta_1) \\ J_2(k_{22}r_1)\cos(2\ \theta_1) \\ J_0(k_{03}r_1)\sin(0\ \theta_1) \\ J_0(k_{03}r_1)\cos(0\ \theta_1) \\ J_3(k_{32}r_1)\sin(3\ \theta_1) \\ J_3(k_{32}r_1)\cos(3\ \theta_1) \\ \vdots \\ J_n(k_{nn'}r_1)\cos(n\ \theta_1) \\ \hline J_0(k_{01}r_1)\sin(0\ \theta_1) \\ J_0(k_{01}r_1)\cos(0\ \theta_1) \\ J_1(k_{11}r_1)\sin(1\ \theta_1) \\ \vdots \\ J_n(k_{nn'}r_1)\cos(n\ \theta_1) \end{pmatrix}$$

Table 1: Bessel Function Zero Crossings

n	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

Table 2: Bessel Function Zero Crossings

$k_{nn'}$	n	n'
2.4048	0	1
5.5201	0	2
8.6537	0	3
11.792	0	4
3.8317	1	1
7.0156	1	2
10.173	1	3
13.324	1	4
5.1356	2	1
8.4172	2	2
11.62	2	3
14.796	2	4
6.3802	3	1
9.761	3	2
13.015	3	3
16.223	3	4
7.5883	4	1
11.065	4	2
14.373	4	3
17.616	4	4

Table 3: Bessel Function Zero Crossings

$k_{nn'}$	n	n'
2.4048	0	1
3.8317	1	1
5.1356	2	1
5.5201	0	2
6.3802	3	1
7.0156	1	2
7.5883	4	1
8.4172	2	2
8.6537	0	3
9.761	3	2
10.173	1	3
11.065	4	2
11.62	2	3
11.792	0	4
13.015	3	3
13.324	1	4
14.373	4	3
14.796	2	4
16.223	3	4
17.616	4	4

Degeneracy of States

$$u(r,t) = \underbrace{\left(g_D(\omega_1 t)\ \dots\ g_D(\omega_m t)\,\middle|\, g_N(\omega_1 t)\ \dots\ g_N(\omega_m t)\right)}_{1\times 2\cdot m} \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & & 0 & 0 \\ \vdots & & & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 1 \end{pmatrix}}_{2\cdot m\times 2\cdot n} \underbrace{\begin{pmatrix} J_0(k_{01} r_1)\sin(0\ \theta_1)\cdot AC_1 \\ J_0(k_{01} r_1)\cos(0\ \theta_1)\cdot AC_2 \\ J_1(k_{11} r_1)\sin(1\ \theta_1)\cdot AC_3 \\ J_1(k_{11} r_1)\cos(1\ \theta_1)\cdot AC_4 \\ J_2(k_{21} r_1)\sin(2\ \theta_1)\cdot AC_5 \\ \vdots \\ J_n(k_{nn'} r_1)\cos(n\ \theta_1)\cdot AC_n \\ \hline J_0(k_{01} r_1)\sin(0\ \theta_1)\cdot AD_1 \\ J_0(k_{01} r_1)\cos(0\ \theta_1)\cdot AD_2 \\ J_1(k_{11} r_1)\sin(1\ \theta_1)\cdot AD_3 \\ J_1(k_{11} r_1)\cos(1\ \theta_1)\cdot AD_4 \\ J_2(k_{21} r_1)\sin(2\ \theta_1)\cdot AD_5 \\ \vdots \\ J_n(k_{nn'} r_1)\cos(n\ \theta_1)\cdot AD_n \end{pmatrix}}_{2\cdot n\times 1}$$

the Degeneracy Matrix $\equiv \mathbb{D}$

$$u(r,t) = \underbrace{\left(g_D(\omega_1 t)\ \dots\ g_D(\omega_m t)\,\middle|\, g_N(\omega_1 t)\ \dots\ g_N(\omega_m t)\right)} \underbrace{\begin{pmatrix} J_0(k_{01} r_1)\sin(0\ \theta_1)\cdot AC_1 + J_0(k_{01} r_1)\cos(0\ \theta_1)\cdot AC_2 \\ J_1(k_{11} r_1)\sin(1\ \theta_1)\cdot AC_3 + J_1(k_{11} r_1)\cos(1\ \theta_1)\cdot AC_4 \\ J_2(k_{21} r_1)\sin(2\ \theta_1)\cdot AC_5 + J_2(k_{21} r_1)\cos(2\ \theta_1)\cdot AC_6 \\ \vdots \\ J_n(k_{nn'} r_1)\sin(n\ \theta_1)\cdot AC_n + J_n(k_{nn'} r_1)\cos(n\ \theta_1)\cdot AC_n \\ \hline J_0(k_{01} r_1)\sin(0\ \theta_1)\cdot AD_1 + J_0(k_{01} r_1)\cos(0\ \theta_1)\cdot AD_2 \\ J_1(k_{11} r_1)\sin(1\ \theta_1)\cdot AD_3 + J_1(k_{11} r_1)\cos(1\ \theta_1)\cdot AD_4 \\ J_2(k_{21} r_1)\sin(2\ \theta_1)\cdot AD_5 + J_2(k_{21} r_1)\cos(2\ \theta_1)\cdot AD_6 \\ \vdots \\ J_n(k_{nn'} r_1)\sin(n\ \theta_1)\cdot AD_{n-1} + J_n(k_{nn'} r_1)\cos(n\ \theta_1)\cdot AD_n \end{pmatrix}}$$

Degeneracy and Sampling

- Due to symmetry in the solution, degenerate states are produced.
- Due to degeneracy, there is a 2-1 ratio of unknowns-knowns.
- Solution: add another sample location

$$\underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \\ \hline \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{pmatrix}}_{2\cdot m\times 1} = \underbrace{\left(\begin{array}{ccccc|ccccc} \Delta_{11} & \Delta_{12} & \Delta_{13} & \cdots & \Delta_{1m} & \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \cdots & \varepsilon_{1m} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & & \Delta_{2m} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & & \varepsilon_{2m} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & & \Delta_{3m} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} & & \varepsilon_{3m} \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ \Delta_{m1} & \Delta_{m2} & \Delta_{m3} & \cdots & \Delta_{mm} & \varepsilon_{m1} & \varepsilon_{m2} & \varepsilon_{m3} & \cdots & \varepsilon_{mm} \\ \hline \varepsilon_{11} & \varepsilon_{21} & \varepsilon_{31} & \cdots & \varepsilon_{m1} & \Delta^{c}_{11} & \Delta^{c}_{12} & \Delta^{c}_{13} & \cdots & \Delta^{c}_{1m} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{32} & & \varepsilon_{m2} & \Delta^{c}_{21} & \Delta^{c}_{22} & \Delta^{c}_{23} & & \Delta^{c}_{2m} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} & & \varepsilon_{m3} & \Delta^{c}_{21} & \Delta^{c}_{32} & \Delta^{c}_{33} & & \Delta^{c}_{3m} \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ \varepsilon_{1m} & \varepsilon_{2m} & \varepsilon_{3m} & \cdots & \varepsilon_{mm} & \Delta^{c}_{m1} & \Delta^{c}_{m2} & \Delta^{c}_{m3} & \cdots & \Delta^{c}_{mm} \end{array}\right)}_{2\cdot m\times 2\cdot m} \underbrace{\begin{pmatrix} J_0(k_{01} r_1)\sin(0\,\theta_1)\cdot AC_1 + J_0(k_{01} r_1)\cos(0\,\theta_1)\cdot AC_2 \\ J_1(k_{11} r_1)\sin(1\,\theta_1)\cdot AC_3 + J_1(k_{11} r_1)\cos(1\,\theta_1)\cdot AC_4 \\ J_2(k_{21} r_1)\sin(2\,\theta_1)\cdot AC_5 + J_2(k_{21} r_1)\cos(2\,\theta_1)\cdot AC_6 \\ \vdots \\ J_n(k_{nn'} r_1)\sin(n\,\theta_1)\cdot AC_{n-1} + J_n(k_n r_1)\cos(n\,\theta_1)\cdot AC_n \\ \hline J_0(k_{01} r_1)\sin(0\,\theta_1)\cdot AD_1 + J_0(k_{01} r_1)\cos(0\,\theta_1)\cdot AD_2 \\ J_1(k_{11} r_1)\sin(1\,\theta_1)\cdot AD_3 + J_1(k_{11} r_1)\cos(1\,\theta_1)\cdot AD_4 \\ J_2(k_{21} r_1)\sin(2\,\theta_1)\cdot AD_5 + J_2(k_{21} r_1)\cos(2\,\theta_1)\cdot AD_6 \\ \vdots \\ J_n(k_{nn'} r_1)\sin(n\,\theta_1)\cdot AD_{n-1} + J_n(k_n r_1)\cos(n\,\theta_1)\cdot AD_n \end{pmatrix}}_{2\cdot m\times 1}$$

Second Location

$$
\underbrace{\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \\ \beta_1 \\ \vdots \\ \beta_m \\ \hline \gamma_1 \\ \vdots \\ \gamma_m \\ \eta_1 \\ \vdots \\ \eta_m \end{pmatrix}}_{4\cdot m \times 1}
=
\underbrace{\left(\begin{array}{cccc|cccc}
\Delta_{11} & \Delta_{1m} & \varepsilon_{11} & \varepsilon_{1m} & & & & \\
\ddots & & \ddots & & & & & \\
\Delta_{m1} & \Delta_{mm} & \varepsilon_{m1} & \varepsilon_{mm} & & 0 & & \\
\varepsilon_{11} & \varepsilon_{m1} & \Delta^{c}_{11} & \Delta^{c}_{1m} & & & & \\
\ddots & & \ddots & & & & & \\
\varepsilon_{1m} & \varepsilon_{mm} & \Delta^{c}_{m1} & \Delta^{c}_{mm} & & & & \\
\hline
& & & & \Delta_{11} & \Delta_{1m} & \varepsilon_{11} & \varepsilon_{1m} \\
& & & & \ddots & & \ddots & \\
& 0 & & & \Delta_{m1} & \Delta_{mm} & \varepsilon_{m1} & \varepsilon_{mm} \\
& & & & \varepsilon_{11} & \varepsilon_{m1} & \Delta^{c}_{11} & \Delta^{c}_{1m} \\
& & & & \ddots & & \ddots & \\
& & & & \varepsilon_{1m} & \varepsilon_{mm} & \Delta^{c}_{m1} & \Delta^{c}_{mm}
\end{array}\right)}_{\text{time-Delta matrix} \quad 4\cdot m \times 4\cdot m}
\underbrace{\left(\begin{array}{cccccccccc|c}
1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & & \\
0 & 0 & 1 & 1 & 0 & 0 & & 0 & 0 & & \\
0 & 0 & 0 & 0 & 1 & 1 & & 0 & 0 & & 0 \\
\vdots & & & & & & \ddots & & \vdots & & \\
0 & 0 & 0 & 0 & 0 & 0 & & 1 & 1 & & \\
\hline
& & & & & & & & & & \begin{matrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & & & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{matrix} \\
& & & & 0 & & & & & &
\end{array}\right)}_{\text{Degeneracy matrix} \quad 4\cdot m \times 4\cdot n}
\underbrace{\left(\begin{array}{c|c}
\begin{matrix} J_0(k_{01} r_1)\sin(0\ \theta_1) \\ J_0(k_{01} r_1)\cos(0\ \theta_1) \\ J_1(k_{11} r_1)\sin(1\ \theta_1) \\ J_1(k_{11} r_1)\cos(1\ \theta_1) \\ J_2(k_{21} r_1)\sin(2\ \theta_1) \\ \ddots \\ J_n(k_{nn'} r_1)\cos(n\ \theta_1) \end{matrix} & 0 \\
\hline
0 & \begin{matrix} J_0(k_{01} r_1)\sin(0\ \theta_1) \\ J_0(k_{01} r_1)\cos(0\ \theta_1) \\ J_1(k_{11} r_1)\sin(1\ \theta_1) \\ J_1(k_{11} r_1)\cos(1\ \theta_1) \\ J_2(k_{21} r_1)\sin(2\ \theta_1) \\ \ddots \\ J_n(k_{nn'} r_1)\cos(n\ \theta_1) \end{matrix} \\
\hline
\begin{matrix} J_0(k_{01} r_2)\sin(0\ \theta_2) \\ J_0(k_{01} r_2)\cos(0\ \theta_2) \\ J_1(k_{11} r_2)\sin(1\ \theta_2) \\ J_1(k_{11} r_2)\cos(1\ \theta_2) \\ J_2(k_{21} r_2)\sin(2\ \theta_2) \\ \ddots \\ J_n(k_{nn'} r_2)\cos(n\ \theta_2) \end{matrix} & 0 \\
\hline
0 & \begin{matrix} J_0(k_{01} r_2)\sin(0\ \theta_2) \\ J_0(k_{01} r_2)\cos(0\ \theta_2) \\ J_1(k_{11} r_2)\sin(1\ \theta_2) \\ J_1(k_{11} r_2)\cos(1\ \theta_2) \\ J_2(k_{21} r_2)\sin(2\ \theta_2) \\ \ddots \\ J_n(k_{nn'} r_2)\cos(n\ \theta_2) \end{matrix}
\end{array}\right)}_{\text{Sample matrix} \quad 4\cdot n \times 2\cdot n}
\underbrace{\begin{pmatrix} AC_1 \\ AC_2 \\ AC_3 \\ AC_4 \\ AC_5 \\ \vdots \\ AC_n \\ \hline AD_1 \\ AD_2 \\ AD_3 \\ AD_4 \\ AD_5 \\ \vdots \\ AD_n \end{pmatrix}}_{2\cdot n \times 1}
$$

The second sampled point $\Longrightarrow$

Calculating the Amplitudes

$$
\underbrace{\begin{pmatrix} AC_1 \\ AC_2 \\ AC_3 \\ AC_4 \\ AC_5 \\ \vdots \\ AC_n \\ \hline AD_1 \\ AD_2 \\ AD_3 \\ AD_4 \\ AD_5 \\ \vdots \\ AD_n \end{pmatrix}}_{2\cdot n\times 1}
=
\underbrace{\left(
\left(\begin{array}{cccccccc|c}
1 & 1 & 0 & 0 & 0 & 0 & \dots & 0\ 0 & \\
0 & 0 & 1 & 1 & 0 & 0 & & 0\ 0 & \\
0 & 0 & 0 & 0 & 1 & 1 & & 0\ 0 & 0 \\
\vdots & & & & & & \ddots & \vdots & \\
0 & 0 & 0 & 0 & 0 & 0 & & 1\ 1 & \\
\hline
 & & & 0 & & & & & \begin{array}{cccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0\ 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \dots & 0\ 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots & 0\ 0 \\ \vdots & & & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1\ 1 \end{array}
\end{array}\right)
\left(\begin{array}{c|c}
\begin{array}{l} J_0(k_{01}\,r_1)\sin(0\ \theta_1) \\ J_0(k_{01}\,r_1)\cos(0\ \theta_1) \\ J_1(k_{11}\,r_1)\sin(1\ \theta_1) \\ J_1(k_{11}\,r_1)\cos(1\ \theta_1) \\ J_2(k_{21}\,r_1)\sin(2\ \theta_1) \\ \ddots \\ J_n(k_{nn'}\,r_1)\cos(n\ \theta_1) \end{array} & 0 \\
\hline
0 & \begin{array}{l} J_0(k_{01}\,r_1)\sin(0\ \theta_1) \\ J_0(k_{01}\,r_1)\cos(0\ \theta_1) \\ J_1(k_{11}\,r_1)\sin(1\ \theta_1) \\ J_1(k_{11}\,r_1)\cos(1\ \theta_1) \\ J_2(k_{21}\,r_1)\sin(2\ \theta_1) \\ \ddots \\ J_n(k_{nn'}\,r_1)\cos(n\ \theta_1) \end{array} \\
\hline
\begin{array}{l} J_0(k_{01}\,r_2)\sin(0\ \theta_2) \\ J_0(k_{01}\,r_2)\cos(0\ \theta_2) \\ J_1(k_{11}\,r_2)\sin(1\ \theta_2) \\ J_1(k_{11}\,r_2)\cos(1\ \theta_2) \\ J_2(k_{21}\,r_2)\sin(2\ \theta_2) \\ \ddots \\ J_n(k_{nn'}\,r_2)\cos(n\ \theta_2) \end{array} & 0 \\
\hline
0 & \begin{array}{l} J_0(k_{01}\,r_2)\sin(0\ \theta_2) \\ J_0(k_{01}\,r_2)\cos(0\ \theta_2) \\ J_1(k_{11}\,r_2)\sin(1\ \theta_2) \\ J_1(k_{11}\,r_2)\cos(1\ \theta_2) \\ J_2(k_{21}\,r_2)\sin(2\ \theta_2) \\ \ddots \\ J_n(k_{nn'}\,r_2)\cos(n\ \theta_2) \end{array}
\end{array}\right)
\right)^{-1}}_{4\cdot m\times 2\cdot n}
\underbrace{\left(\begin{array}{cc|cc}
\Delta_t & \varepsilon_t & & \\
\varepsilon_t^{\dagger} & \Delta_t^{c} & & 0 \\
\hline
 & 0 & \Delta_t & \varepsilon_t \\
 & & \varepsilon_t^{\dagger} & \Delta_t^{c}
\end{array}\right)^{-1}}_{4\cdot m\times 4\cdot m}
\underbrace{\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \\ \beta_1 \\ \vdots \\ \beta_m \\ \hline \gamma_1 \\ \vdots \\ \gamma_m \\ \eta_1 \\ \vdots \\ \eta_m \end{pmatrix}}_{4\cdot m\times 1}
$$

Drum Solution in Short-hand

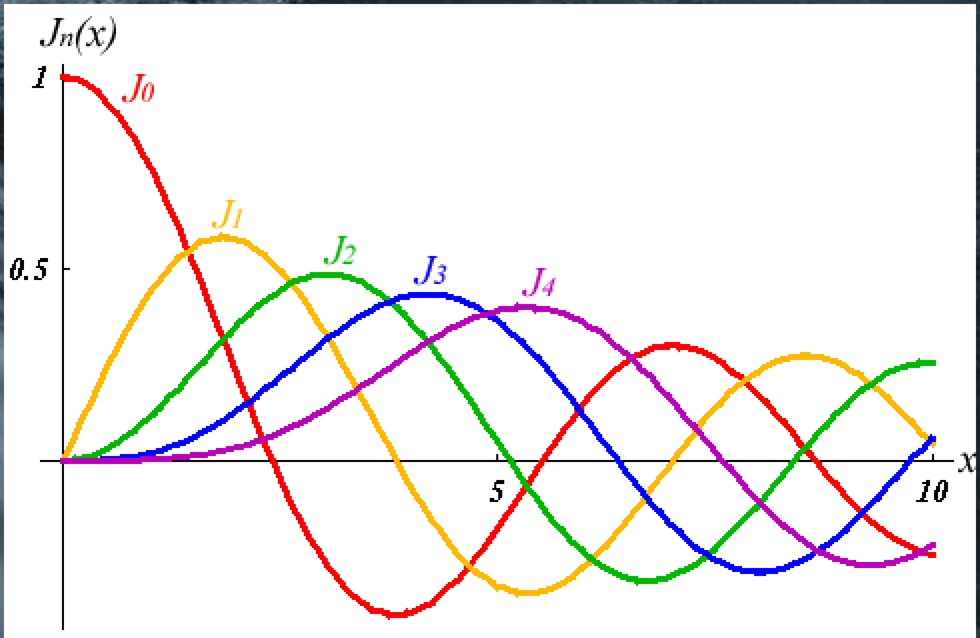
- Number of sample locations “squares-off” the degeneracy matrix, allowing the system to be solvable.

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \eta \end{pmatrix} = \begin{pmatrix} \Delta_t & O \\ O & \Delta_t \end{pmatrix} \left[\begin{pmatrix} \mathbb{D} \end{pmatrix} \begin{pmatrix} f_{D1} & O \\ O & f_{D1} \\ f_{D2} & O \\ O & f_{D2} \end{pmatrix} \right] \begin{pmatrix} AC \\ AD \end{pmatrix}$$

$$\begin{pmatrix} AC \\ AD \end{pmatrix} = \left[\begin{pmatrix} \mathbb{D} \end{pmatrix} \begin{pmatrix} f_{D1} & O \\ O & f_{D1} \\ f_{D2} & O \\ O & f_{D2} \end{pmatrix} \right]^{-1} \begin{pmatrix} \Delta_t & O \\ O & \Delta_t \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \eta \end{pmatrix}$$

Drum Problems

- n=0 sin(nθ) term needs to be removed, breaking the 2-1 ratio to less than 2-1.
- Test method against drum simulation, with known inputs to the amplitudes, A_n.
- Further degeneracies exist due to closeness of k-eigenvalues.



Chesapeake Bay Problems

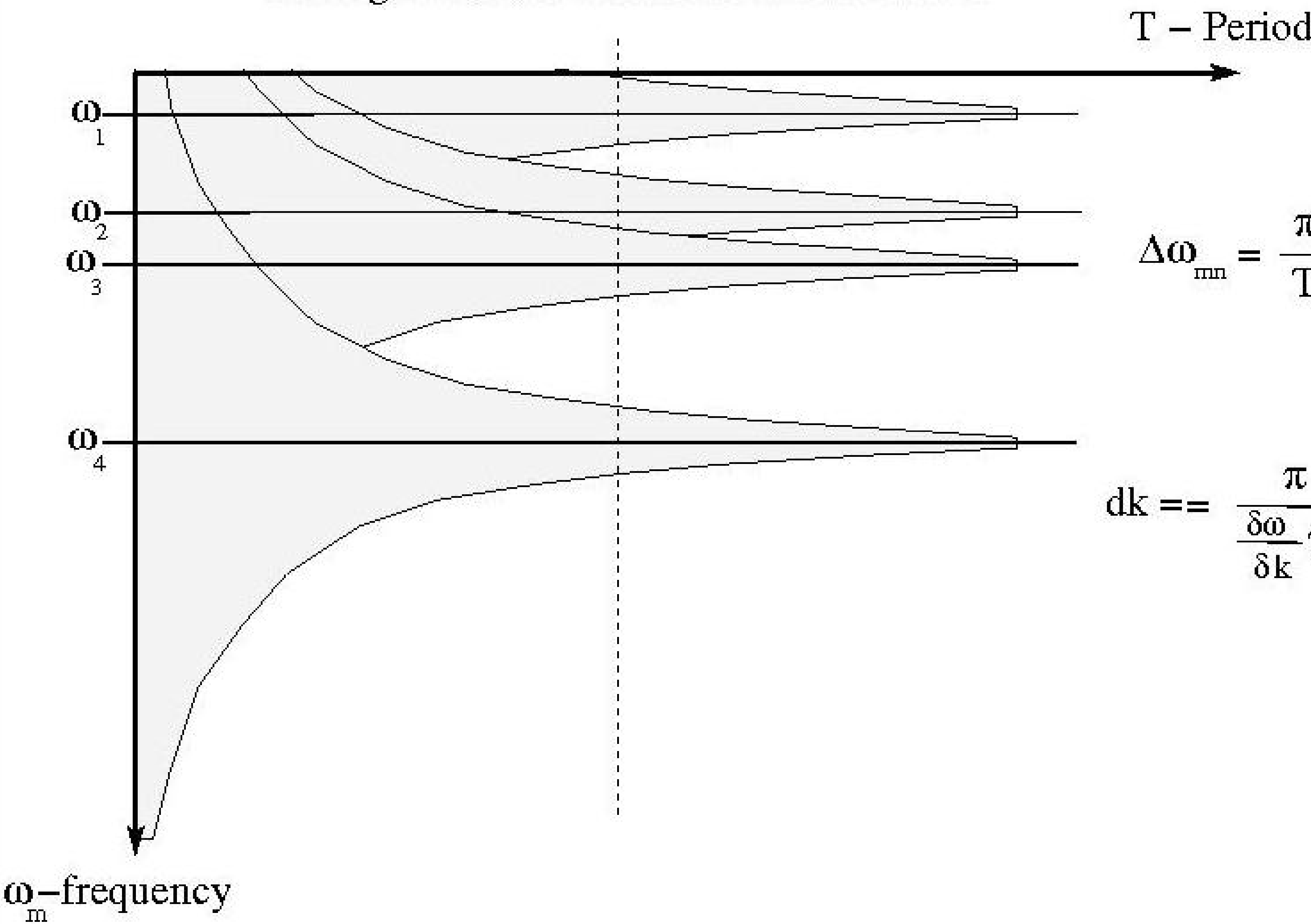
- No known analytic solution to the Bay

  (ie. No analytic hints as to any degeneracy)
- At a given samples location, calculate the projections, (α_m, β_m) for a range of (ω_m, T).
- Observe the patterns of ω_m and compare the sequence to k_n's.
- Guess the dispersion relationship (map from k_n to ω_m

Degeneracy of States

- From observables at frequencies ω_m, observe projection changes as the period,T, is changed.
- Select an appropriate period, T.
- Construct Degeneracy matrix based on best guess of dispersion relation as well as k-eigenvalues density.

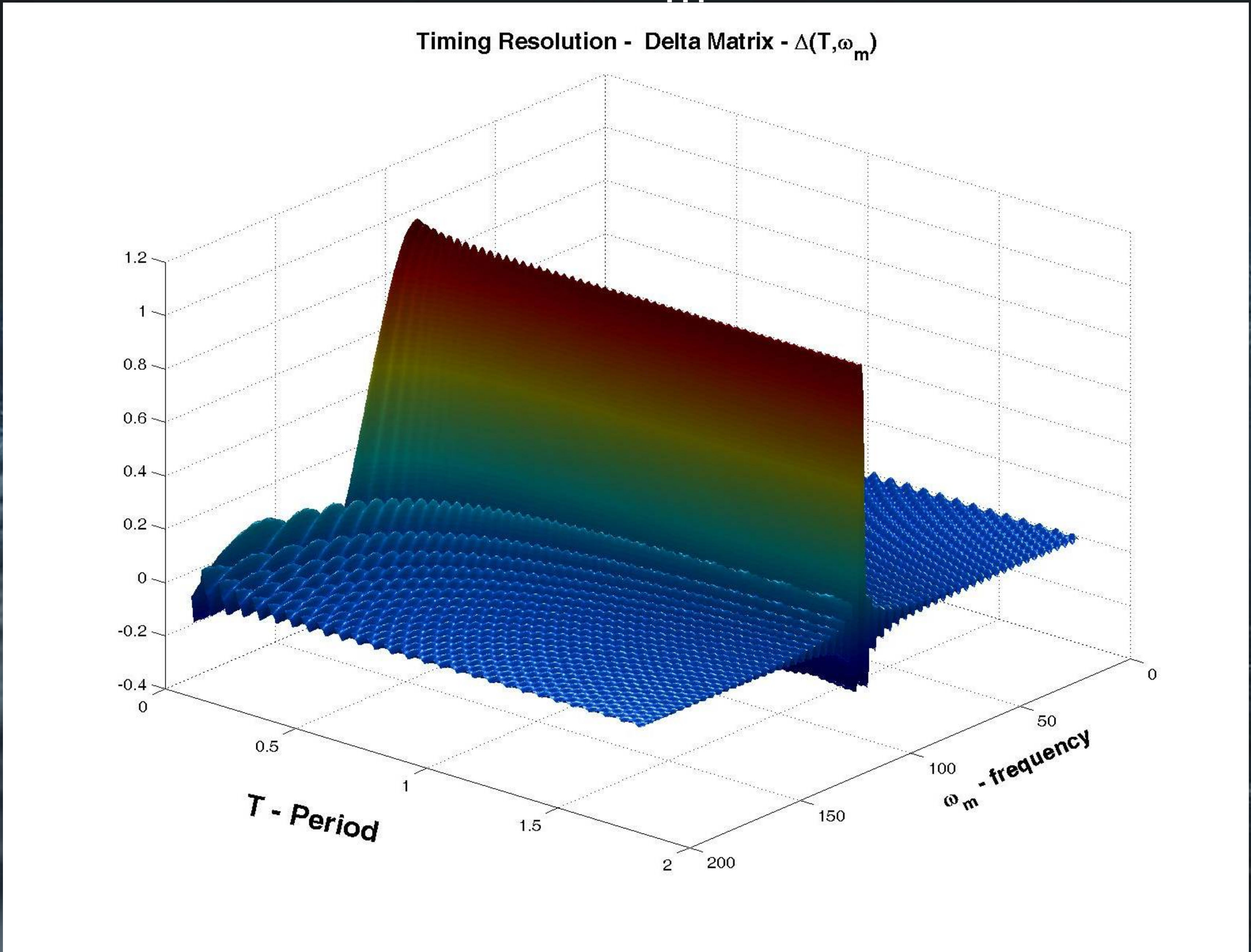
Timing Resolution Nature of Delta Matrix



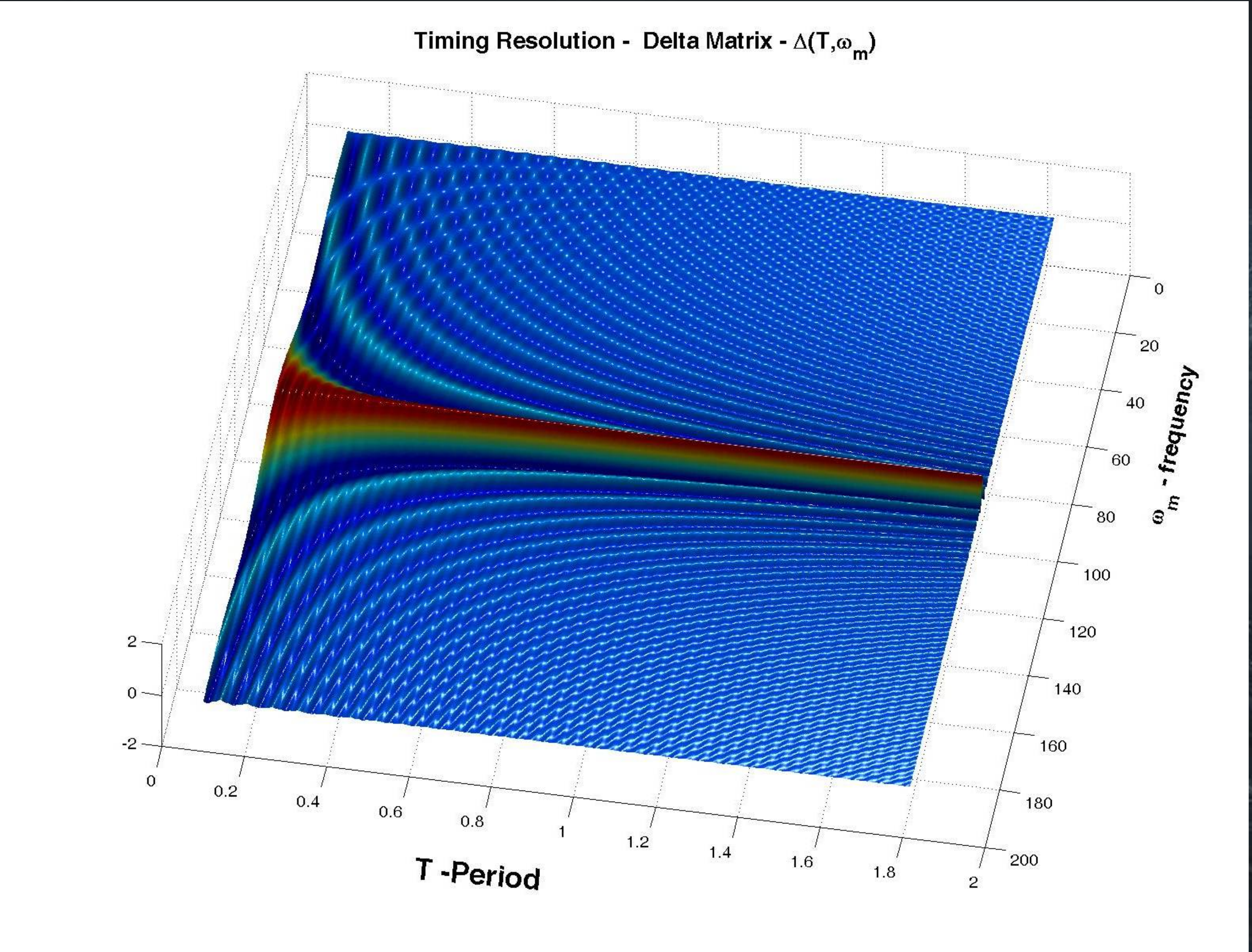
$$\Delta\omega_{mn} = \frac{\pi}{T}$$

$$dk == \frac{\pi}{\frac{\delta\omega}{\delta k}T}$$

Delta(ω_m,T)



Delta(ω_m,T)



Future Plans:

- Run thru the toy model (Drum).
- Add source terms to Chesapeake Bay model.
- Add Delta(spatial) matrix.

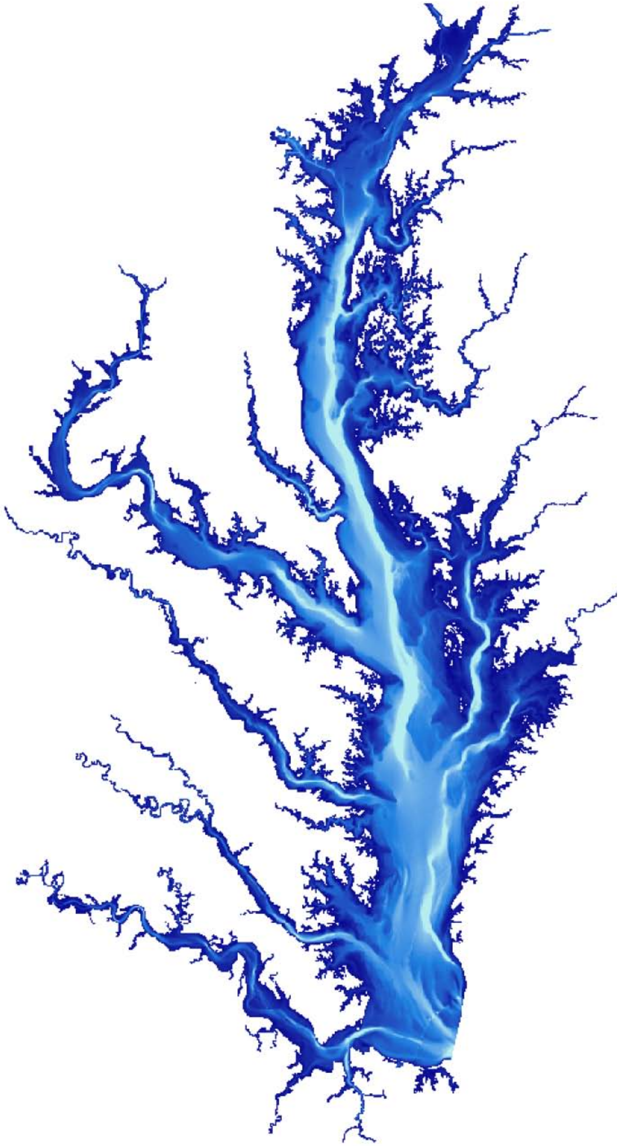
Acknowledgements

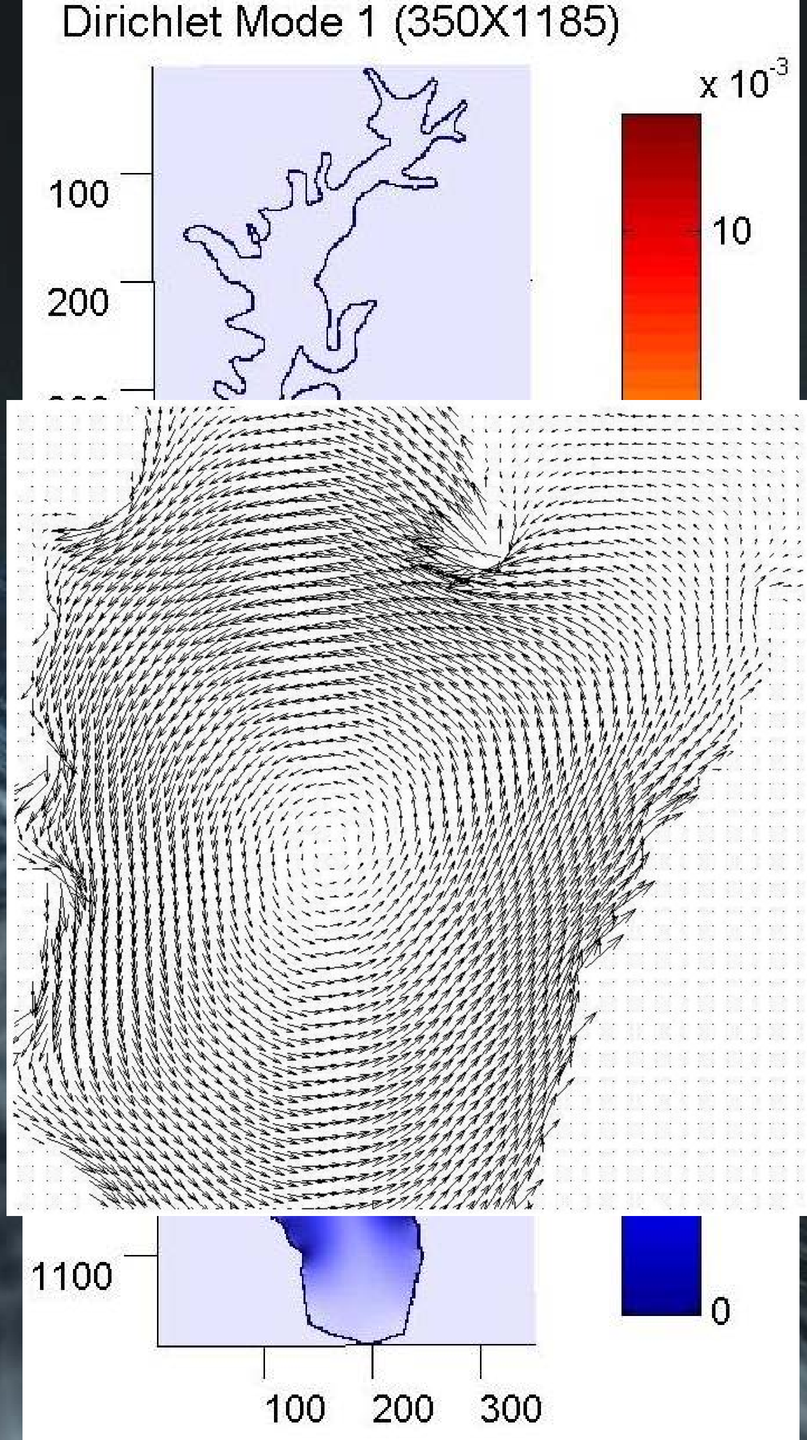
- Student: MIDN 1/C Grant Hundley
- Collaborator: Reza Malek-Madani
- James W. Kinnear (USN – Ret.)

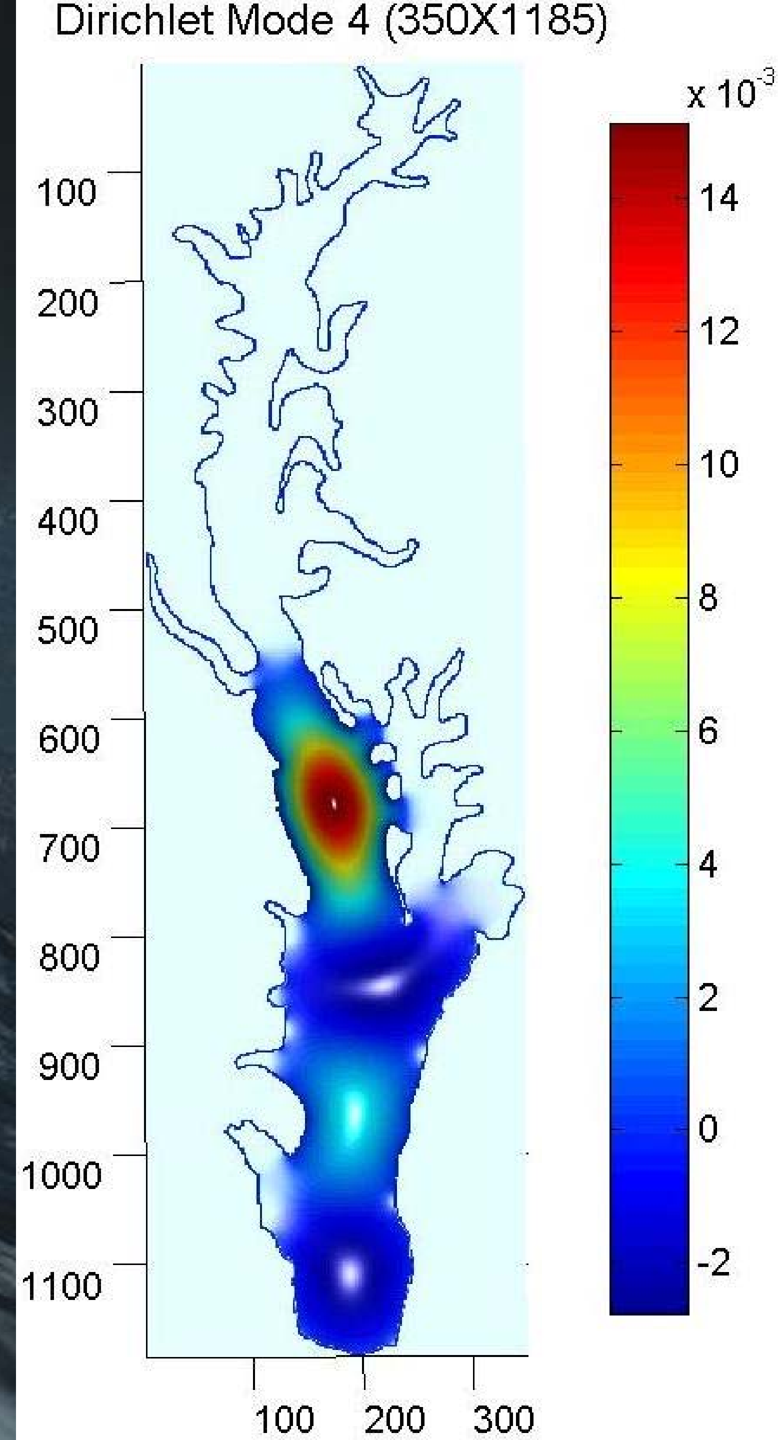
Chesapeake Bay Problem

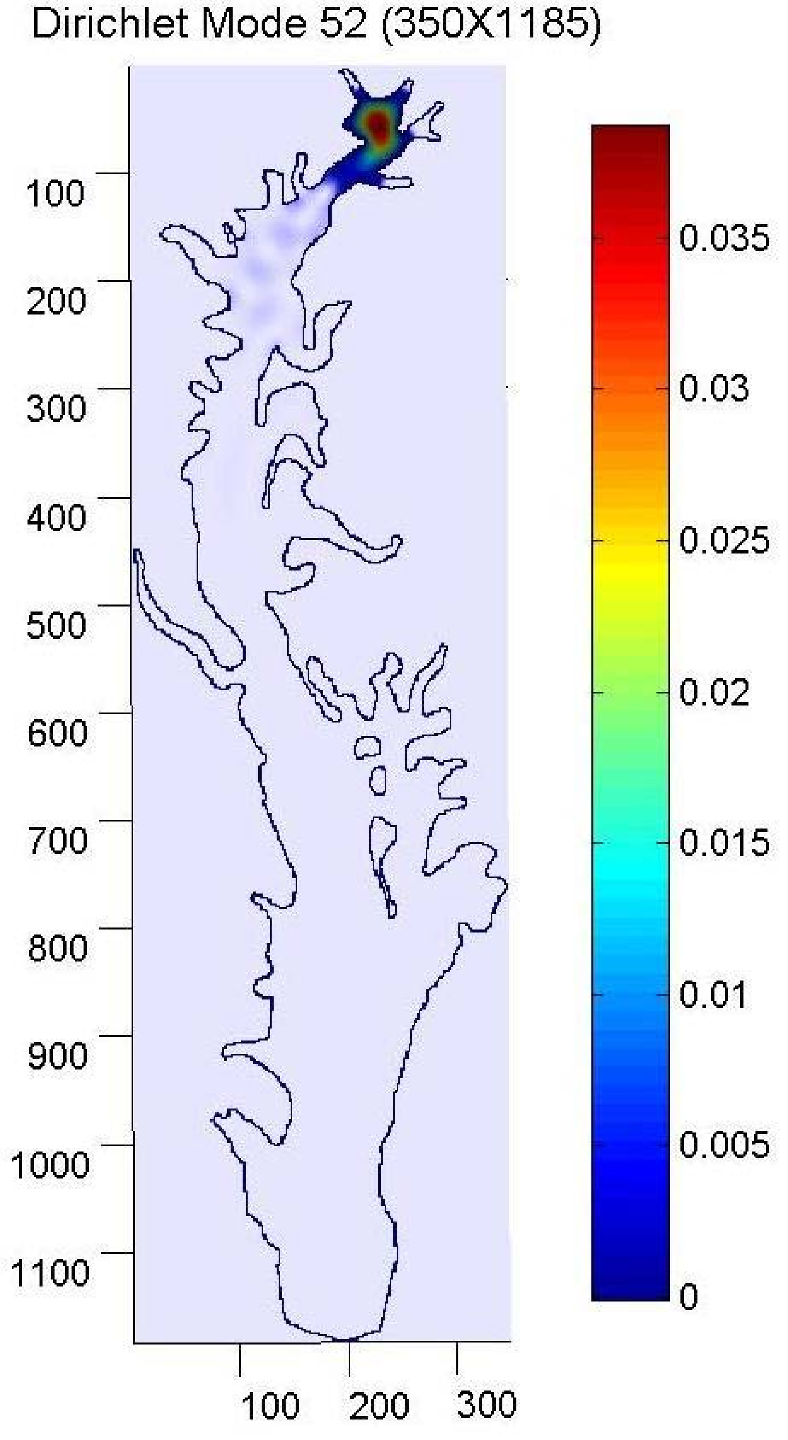
- Take data at stations around the Bay, collecting time-series of vector flows.
- How many stations are needed to provide enough data to fully calculate the modes?

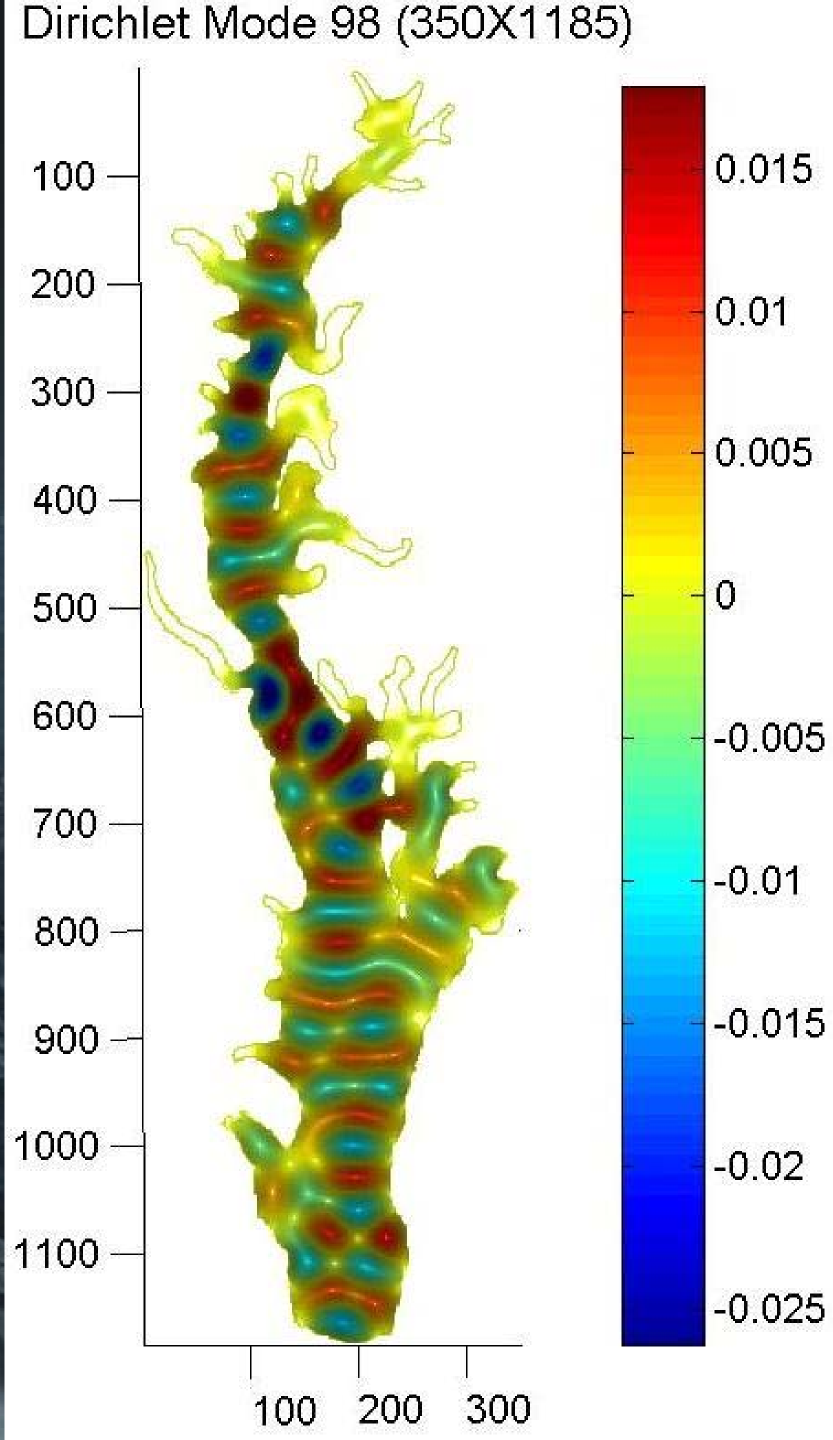
Image Processing of the Chesapeake



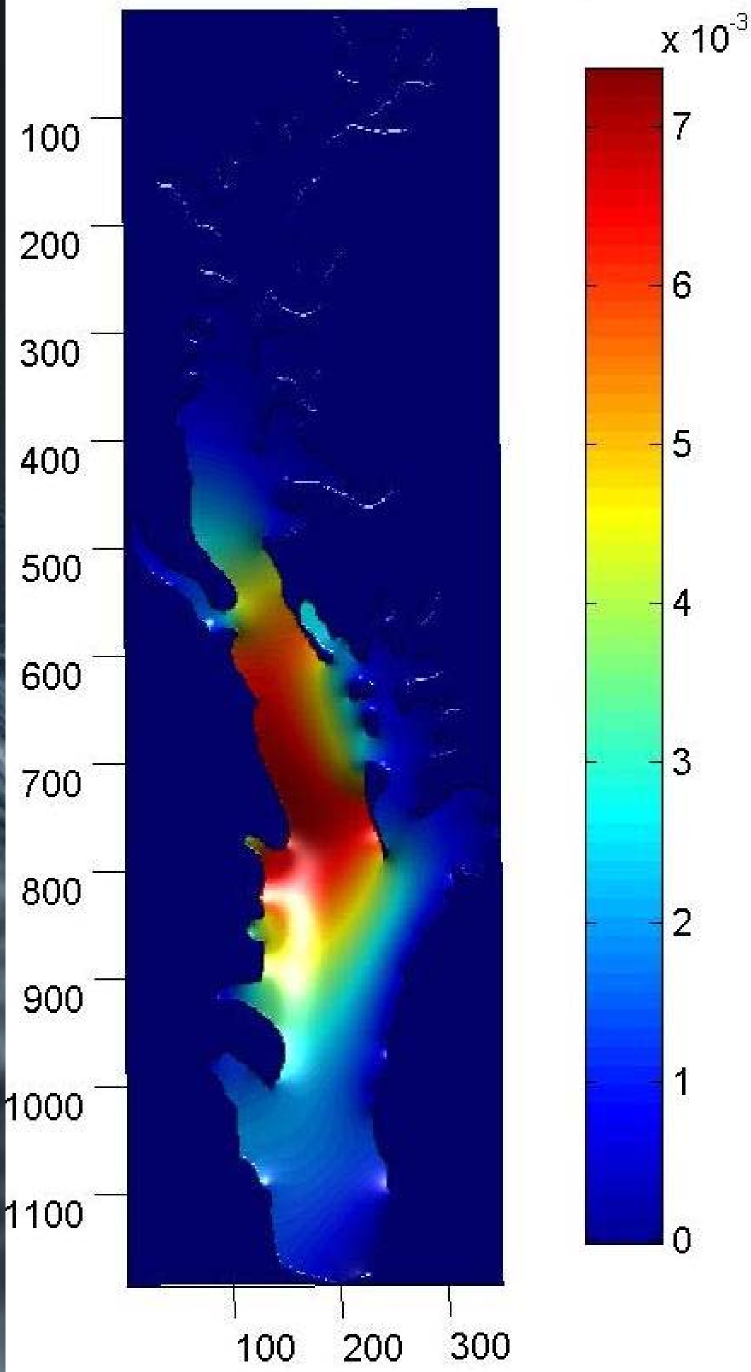

Dirichlet Mode 1 (350X1185)
x 10-3
10
0
100
200
1100
100
200
300


Dirichlet Mode 4 (350X1185)
x 10-3
14
12
10
8
6
4
2
0
-2
100
200
300
400
500
600
700
800
900
1000
1100
100
200
300

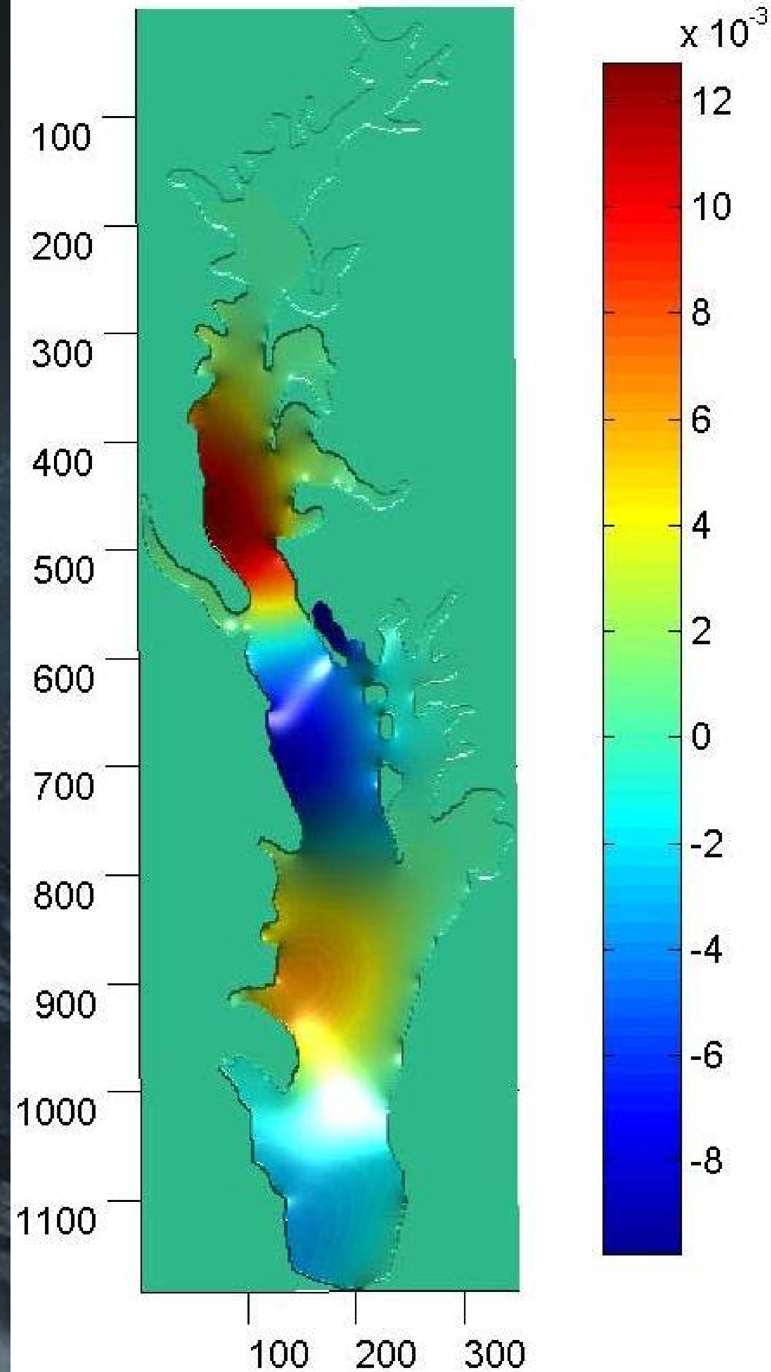

Dirichlet Mode 52 (350X1185)
100
200
300
400
500
600
700
800
900
1000
1100
100 200 300
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0

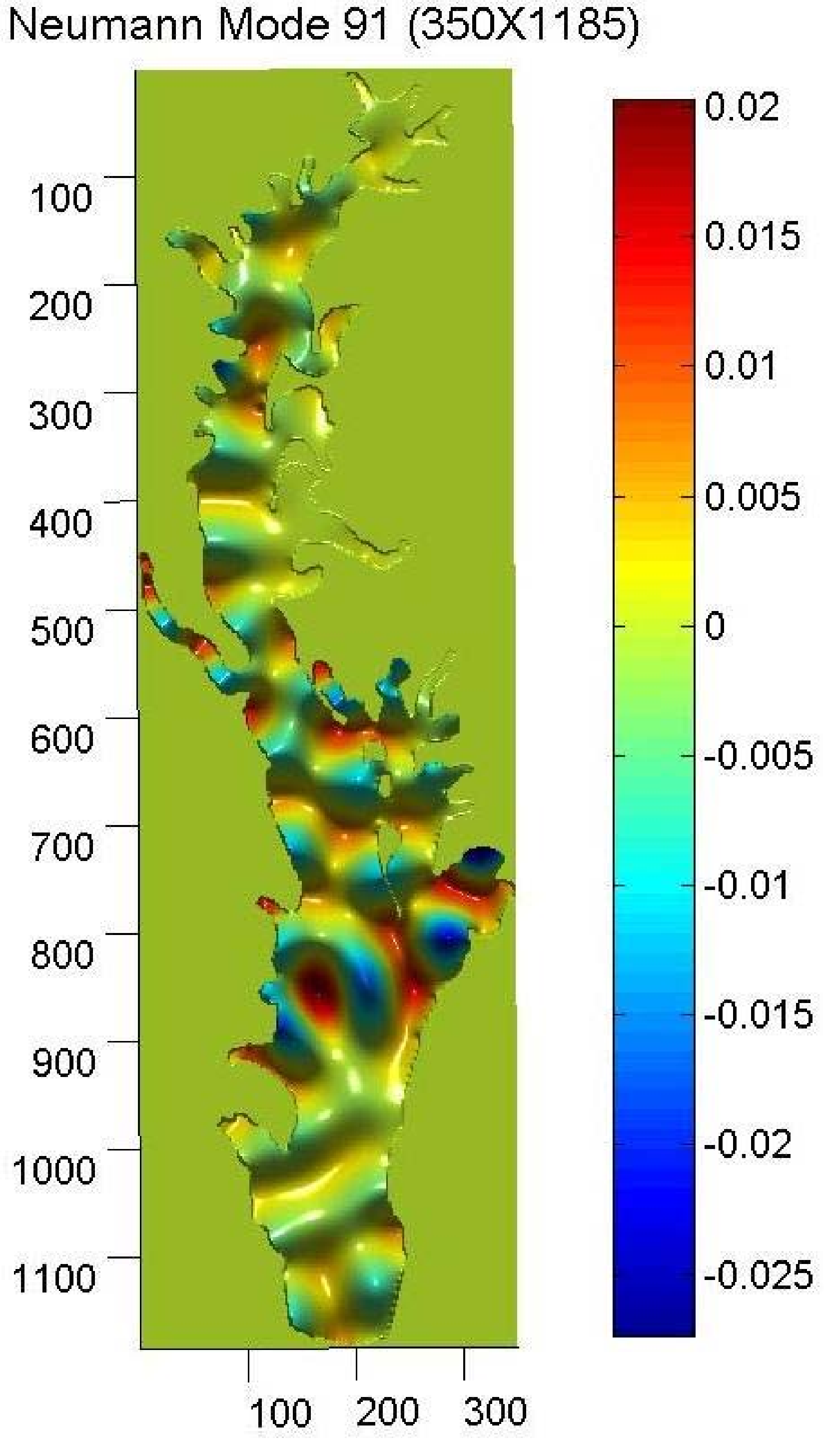

Dirichlet Mode 98 (350X1185)
100
200
300
400
500
600
700
800
900
1000
1100
100 200 300
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025

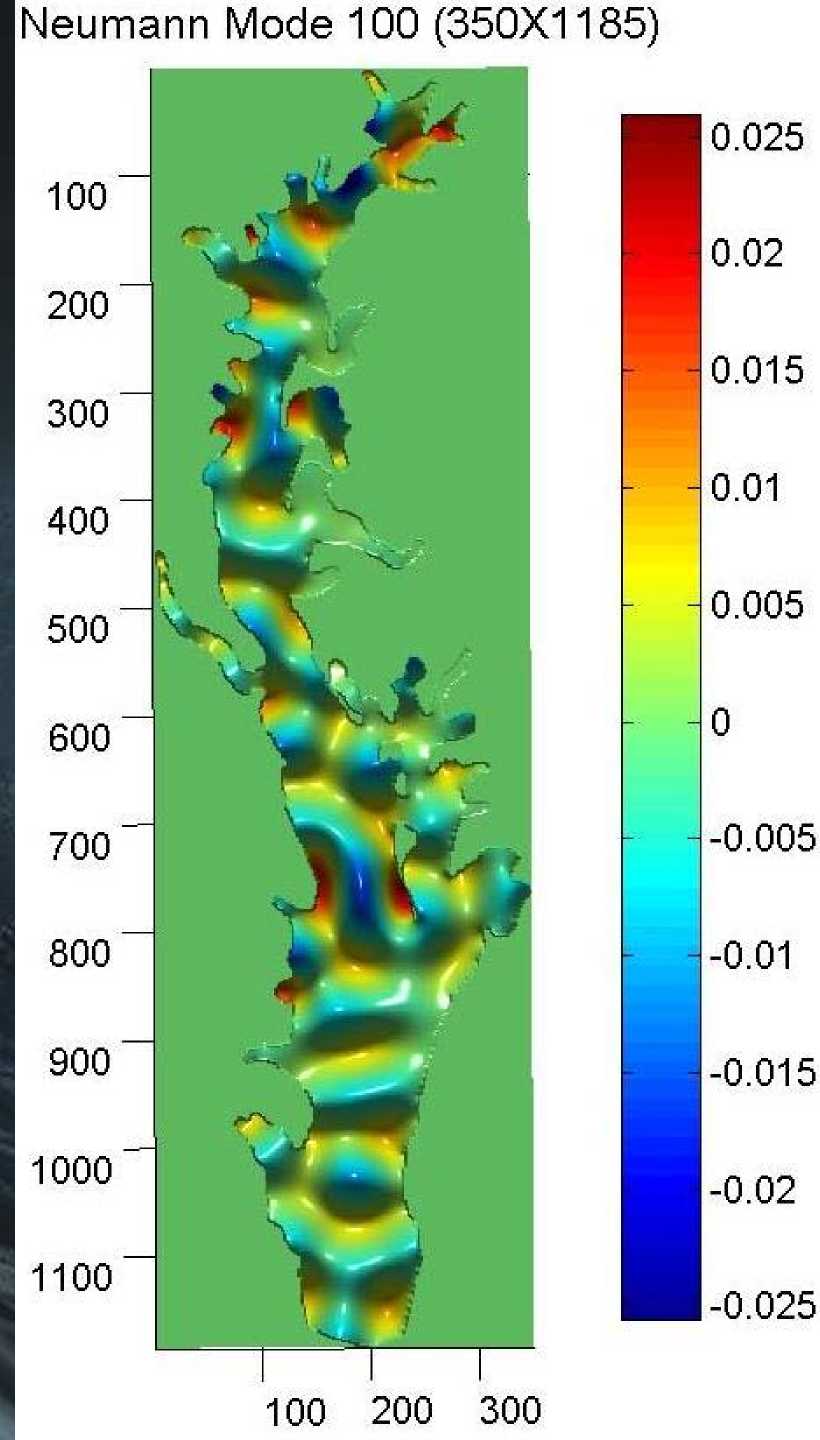
Neumann Mode 1 (350X1185)



Neumann Mode 4 (350X1185)




Neumann Mode 91 (350X1185)
100
200
300
400
500
600
700
800
900
1000
1100
100
200
300
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025


Neumann Mode 100 (350X1185)
100
200
300
400
500
600
700
800
900
1000
1100
100
200
300
0.025
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025

Advantages

- Eigenmodes fill domain (space), suggest future behavior