Stability Analysis of ALE-Methods for Advection-Diffusion Problems

A. Weddemann* and V. Thümmler
Bielefeld University, Department of Mathematics
*Universitätsstr. 25, 33615 Bielefeld, Germany, weddeman@physik.uni-bielefeld.de

Abstract: ALE-methods are frequently used to solve systems of partial differential equations (PDEs) on moving domains. The main idea of these methods is to incorporate the time evolution of the domain into the equations. However, the motion of the domain with respect to time induces convective fluxes in the resulting equations. These can lead to stability problems of the numerical method if they become too large. In this paper we show that these difficulties occur already in very simple systems. We discuss the stability properties of the implicit Euler method applied to a linear advection-diffusion problem on a moving domain and propose several ways of dealing with the stability problems. Furthermore, we compare the direct implementation of the ALE-equations via a weak formulation with the predefined ALE-mode of COMSOL Multiphysics.

Keywords: weak formulation**,** numerical stability, linear advection-diffusion problem, ALE-formulation

1. Introduction

ALE-methods are frequently used to model systems where the physical domain changes with respect to time. Common examples can be found in the field of fluid-structure interactions [1] where the domain movement is due to the force the fluid exerts on a solid object. These systems have gained a lot of interest for example to describe blood flows in haemodynamics [2] or for modeling surfaces or interfaces of fluids.

However, the stability properties of ALE-methods are not well understood in the context of FEM-schemes. In case of finite volume or finite difference-schemes a connection between the numerical stability and the geometrical conservation laws has been found by Farhat et al. [3, 4]. A similar result for FEM-schemes is still missing, although in [2] one can find some results in that direction. Especially, a time step restriction for the implicit Euler method is used for integration in time. However, the time evolution of the error has not been examined there.

In the present work, we investigate the stability properties of the ALE-method applied to a linear advection-diffusion problem on a moving domain. We discuss an implementation via a weak formulation and compare this with the use of the predefined ALE-mode implemented in COMSOL Multiphysics. The experiments show that numerical instabilities become a problem, if the convective fluxes get high in comparison to the diffusive contributions. We suggest different possibilities to deal with these problems.

2. Governing equations

We consider a parabolic PDE

$$\frac{\partial u}{\partial t}(x,t) + \mathcal{L}[u](x,t) = f(x,t) \qquad (1.1)$$

on a bounded, finite domain $\Omega \subset \mathbb{R}^n$ that evolves on a finite time interval $I = [t_0, T]$, where $\mathcal{L}[u]$ is an elliptic differential operator and

$$f \in L^2(\Omega \times I, \mathbb{R}^n)$$

denotes an external forcing term. Let Ω_0 be the domain configuration at time $t = t_0$. We will refer to this as initial or reference configuration. To describe the domain movement we introduce ALE-functions

$$a : \Omega_0 \times I \to \mathbb{R}^n$$
$$(\xi, t) \mapsto a(\xi, t)$$

and $\quad \alpha : \hat{\Omega} \to \Omega_0$

$$(x,t) \mapsto \alpha(x,t)$$

with $\quad a(\alpha(x,t),t) = x$

where $\quad \hat{\Omega} = \{(x,t), x \in \Omega_t, t \in I\}$

and $\quad \Omega_\tau = a(\Omega_0, \tau)$.

Ω_τ denotes the domain configuration at time $\tau \in [t_0, T]$, where $\hat{\Omega}$ is the set of space-time-points. For fixed $t \in I$ we expect the function $a(\cdot, t)$ to be continuous and invertible with continuous inverse. Furthermore, for fixed $\xi \in \Omega_0$ the function $a(\xi, \cdot)$ is assumed to be differentiable.

Multiplying equation (1.1) with an appropriate smooth test function ψ and integrating over the domain Ω_t, we obtain the weak formulation

$$\int_{\Omega_t} \psi(x,t) \cdot \frac{\partial u}{\partial t}(x,t) dx + \int_{\Omega_t} \psi(x,t) \cdot \mathcal{L}[u](x,t) dx = \int_{\Omega_t} \psi(x,t) \cdot f(x,t) dx \qquad (1.2)$$

Mapping the domain Ω_t back onto the initial configuration Ω_0, we can reformulate equation (1.2) with respect to the reference domain Ω_0.

$$\int_{\Omega_0} \hat{\psi}(\xi,t) \cdot \frac{\partial \hat{u}}{\partial t}(\xi,t) \cdot \det\left(\frac{\partial a}{\partial \xi}\right)(\xi,t) d\xi + \int_{\Omega_0} \hat{\psi}(\xi,t) \cdot \widehat{\mathcal{L}[u]}(\xi,t) \cdot \det\left(\frac{\partial a}{\partial \xi}\right)(\xi,t) d\xi = \int_{\Omega_0} \hat{\psi}(\xi,t) \cdot \hat{f}(\xi,t) \cdot \det\left(\frac{\partial a}{\partial \xi}\right)(\xi,t) d\xi, \qquad (1.3)$$

where we set

$$\hat{v}(\xi,t) = v(a(\xi,t),t).$$

We will use further on the fact that the test functions $\hat{\psi}$ can be chosen to be independent of time t (see [2]). (1.3) is the weak form of the ALE-formulation of the original problem (1.1). Note, that the domain movement has been incorporated into the equations.

3. Investigated model

Our model system is a linear advection-diffusion problem. Let $D > 0$ and $t_0 = 0$. Then we address the following problem:

Find a function $u : \hat{\Omega} \to \mathbb{R}$, such that

$$\begin{aligned} &\frac{\partial u}{\partial t} - D\Delta_x u = f && \text{in } \hat{\Omega} \\ &u(x,t) = 0 && \text{for } x \in \partial\Omega_t,\ t \in I, \\ &u(\xi,0) = 16 \cdot \xi_1(1-\xi_1) \cdot \xi_2(1-\xi_2) && \text{for } \xi \in \Omega_0. \end{aligned} \qquad (2.1)$$

The reference domain Ω_0 is the unit square $[0,1] \times [0,1]$ and the domain movement is given by

$$a_i(\xi,t) = \xi_i(2-\cos(10\pi t))$$

and
$$\alpha_i(x,t) = \frac{x_i}{2-\cos(10\pi t)}. \qquad (2.2)$$

The motion of the domain is illustrated in Fig. 1. For the following special choice of the force term f

$$\begin{aligned} \hat{f}(\xi,t) &= f(a(\xi,t),t) \\ &= 40\pi\cos(5\pi t) \cdot \xi_1(1-\xi_1) \cdot \xi_2(1-\xi_2) \\ &+ \frac{32D \cdot (1+0,5\sin(5\pi t))}{(2-\cos(10\pi t))^2} \\ &\cdot (\xi_1(1-\xi_1) + \xi_2(1-\xi_2)) \\ &- \frac{160\pi(1+0,5\sin(5\pi t)) \cdot \sin(10\pi t)}{2-\cos(10\pi t)} \\ &\cdot \xi_1\xi_2(2-3\xi_1-3\xi_2+4\xi_1\xi_2) \end{aligned} \qquad (2.3)$$

the exact solution of (2.1) is given by

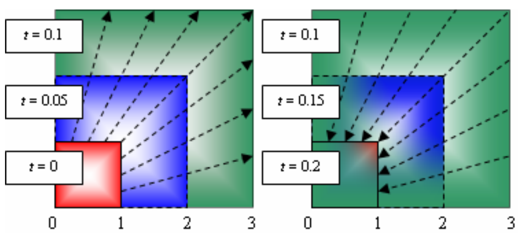

Fig. 1: Domain evolution resulting from (2.2)

$$\hat{u}(\xi,t) = 16\left(1+\frac{1}{2}\sin(5\pi t)\right)$$

$$\cdot\xi_1(1-\xi_1)\cdot\xi_2(1-\xi_2) \qquad (2.4)$$

We further define the differential operator by

$$\mathcal{L}[u](x,t) = -\Delta_x u(x,t)\,.$$

Thus, the weak formulation (1.3) of problem (2.1) can be rewritten using integration by parts as

$$(2-\cos(10\pi t))\int_{\Omega_0}\hat{\psi}(\xi)\frac{\partial\hat{u}}{\partial t}(\xi,t)d\xi$$

$$-\frac{D}{2-\cos(10\pi t)}\int_{\Omega_0}\sum_i\frac{\partial\hat{\psi}(\xi)}{\partial\xi_i}\frac{\partial\hat{u}(\xi,t)}{\partial\xi_i}d\xi$$

$$-10\pi\sin(10\pi t)\int_{\Omega_0}\hat{\psi}(\xi)\sum_i\xi_i\frac{\partial\hat{u}}{\partial\xi_i}d\xi$$

$$=(2-\cos(10\pi t))\int_{\Omega_0}\hat{\psi}(\xi)\cdot\hat{f}(\xi,t)d\xi \qquad (2.5)$$

Equation (2.5) is implemented in COMSOL Multiphysics via weak equation modeling.

4. Simulation with COMSOL

To solve problem (2.1) we use quadratic Lagrangian elements and a finite element mesh with a maximum element size scaling factor of 1 and an element growth rate of 1.3. Furthermore, we choose the maximum element size at the boundaries not to exceed 0.02. The resulting mesh is refined once with a regular refinement method. The domain discretization which we use for the calculation is shown in Fig. 2. It consists of 4098 triangular elements leading to a total number of 8397 degrees of freedom.

The solution of equation (2.5) is carried out for the diffusion constants $D_1 = 0.01$ and $D_2 = 1$. Further numerical results can be found in [5].

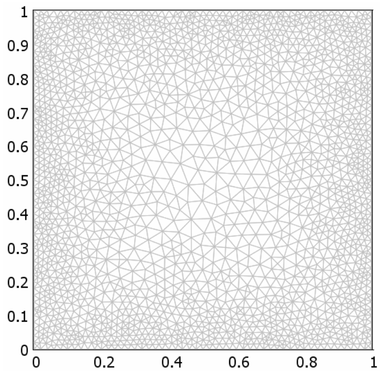

Fig. 2: Finite element mesh used for the simulations

To investigate the dependency of the numerical stability of the implicit Euler method as the time integrator (i.e. BDF of order 1) on the time step size Δt we use the following values

$$\{\Delta t = 1/(20k), k = 1,2,...,15\}\,.$$

To enforce the solver to use exactly these values we set a high absolute and relative tolerance. The results of the numerical simulation are shown in Fig. 3a) and Fig. 3b).

5. Discussion

Even at very small time step sizes there exists a critical time, when the numerical solution strongly deviates from the exact one. To analyze this behavior further we choose a step size of Δt = 1/300. Expanding the simulation time over two periods it can be found that the observed behavior occurs periodically (see Fig. 4). The origin of the problem is the movement of the domain. Analyzing the ALE-expressions (2.2) it can easily be found that the critical time is close to the time when the domain size starts to decrease.

If the difference between numerical and exact solution is analysed at t = 0.24, a high error close to the right and the upper boundary can be observed. To avoid this behaviour we can use different methods:

a) improve the resolution of the time integration scheme by either further refining the step size or using a higher BDF-order
b) use a finer mesh close to the boundaries

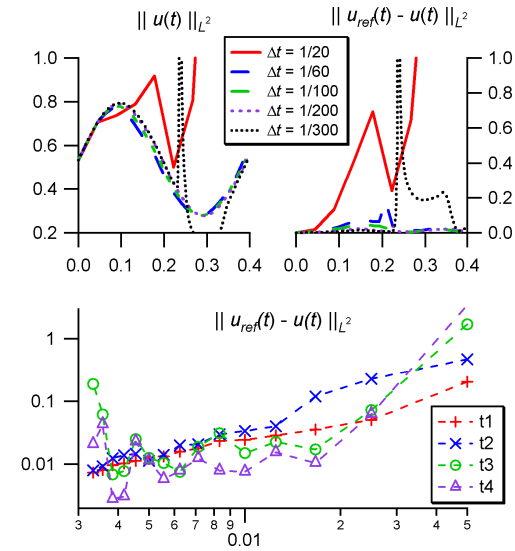

Fig. 3a): Simulation results for D = 0.01: the upper left panel shows the L^2-norm of the numerical solution for different step sizes. The L^2-norm of the difference between the simulation result and the exact solution is plotted in the upper right panel. On the bottom panel the L^2-error at different times is shown for different step sizes.

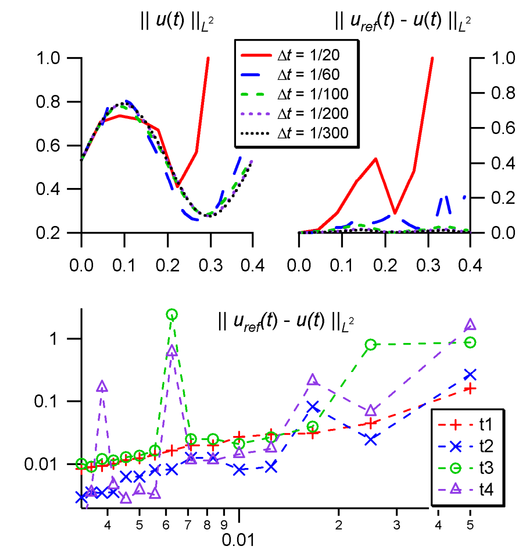

Fig. 3b): Simulation results for D = 1: the upper left panel shows the L^2-norm of the numerical solution for different step sizes. The L^2-norm of the difference between the simulation result and the exact solution is plotted in the upper right panel. On the bottom panel the L^2-error at different times is shown for different step sizes.

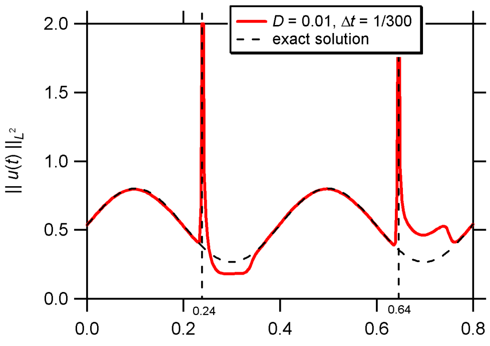

Fig. 4a): Time evolution of the L^2-norm of the numerical solution for two periods: the peaks in the solution show up periodically

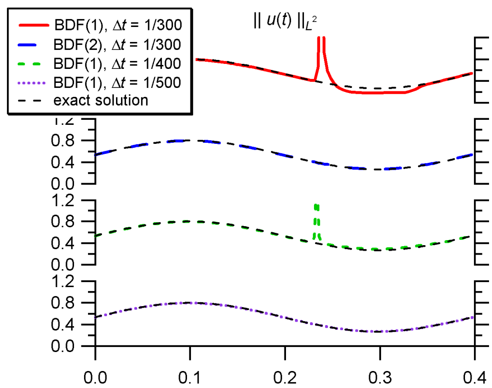

Fig. 4b): Time evolution of the L^2-norm of the numerical solution for different solver parameters

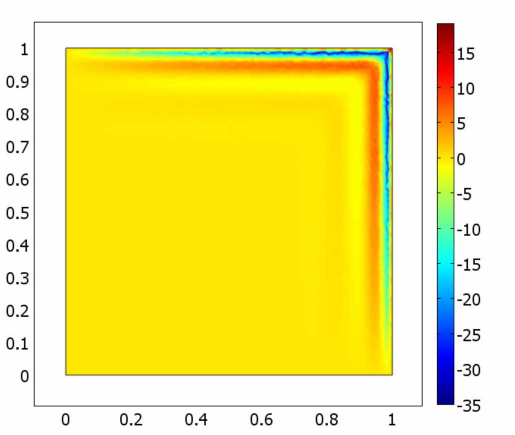

Fig. 5a): Pointwise difference of numerical and exact solution for $D = 0.01$, $\Delta t = 1/300$ at $t = 0.24$

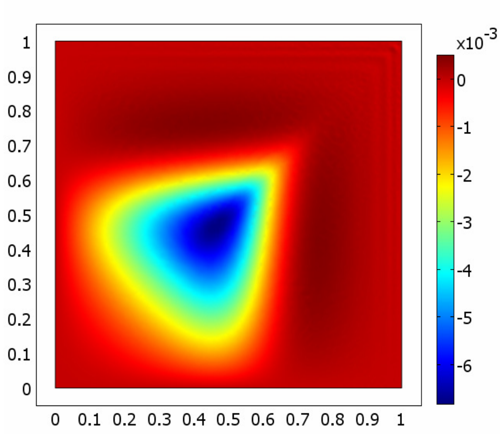

Fig. 5b): Pointwise difference of numerical and exact solution for $D = 0.01$, $\Delta t = 1/500$ at time $t = 0.24$.

If we try to improve the quality of the solution with the help of a method described in a), we make the following observations: Strong deviations from the exact solution can still be found at very small step sizes (Δt = 1/400). Nevertheless, they vanish if the step size is further decreased (Δt = 1/500, see Fig. 4b)). The large errors close to the moving boundaries disappear as shown in Fig. 5b). If we apply time integration schemes of higher order this also leads to an improvement of the result.

To investigate the influence of the spatial discretization, the simulations are carried for different meshes, where the minimal element size at the right and the upper boundary are decreased. The following cases are tested:

min. element size at upper and right boundary	Number of elements	degrees of freedom
0.02	4098	8397
0.015	5194	10625
0.01	7507	15320
0.005	15140	30795
0.002	38770	78671
0.001	79252	160649

Tab. 1: Different meshes applied to equation (2.5)

Apart from a very strong increase of the degrees of freedom resulting in long computational times even at a very fine domain resolution (max. element size = 0.001 at upper and right boundary), small deviations from the exact solutions can still be found (see Fig. 6).

6. Predefined ALE-mode

System (2.1) is solved with the predefined ALE-mode of COMSOL Multiphysics. Here the weak formulation solved by COMSOL is of a slightly different form than equation (2.5). Although the analytic expressions are equivalent, we have found that the numerical results strongly differ from each other.

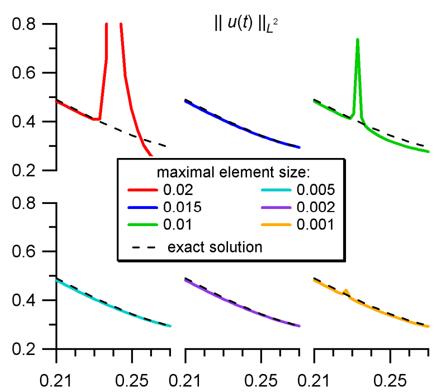

Fig. 6: Behaviour of L^2-norm of the numerical solution close to the critical time for different meshes

The weak equations used for the discretization are given by

$$\int_{\Omega_t} \psi(x,t) \frac{\partial u}{\partial t}(x,t)\,dx$$
$$-D \sum_i \int_{\Omega_t} \frac{\partial \psi(x)}{\partial x_i} \frac{\partial u(x,t)}{\partial x_i}\,dx$$
$$= \int_{\Omega_t} \psi(x,t) \cdot f(x,t)\,dx \qquad (6.1)$$

where $\frac{\partial}{\partial x_i} = \sum_j \frac{\partial \alpha_j(x,t)}{\partial x_i} \frac{\partial}{\partial \xi_j}$

with the ALE-function α as defined in (2.2).
On the continuous level (6.1) is equivalent to (2.5). However, the two formulations lead to different discrete versions of the original problem (2.1), resulting in different numerical solutions for the set of parameters discussed above. The results, using again the mesh shown in Fig. 2, are shown in Fig. 7. We find that even for very small step sizes the numerical solution still strongly deviates from the exact one at specific points in time. Furthermore we now observe several peaks instead of just one. Note that the peak at about $t = 0.24$ which was visible also for the calculation via the weak formulation can be found here as well.

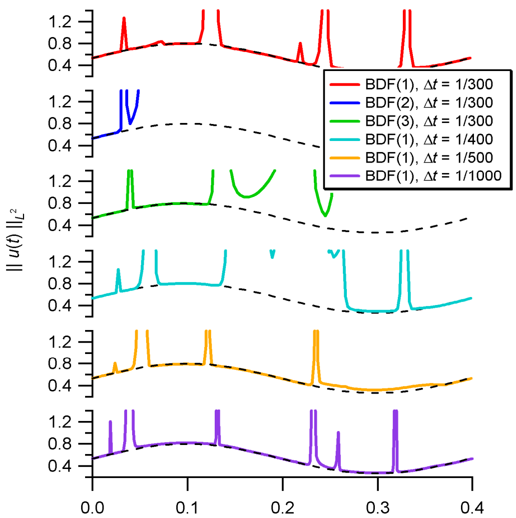

Fig. 7: Solutions obtained by predefined ALE-mode for different parameters

Furthermore, it may be interesting to notice, that in case of modeling with equation (6.1), an increase of the BDF-order does not lead to an increase of accuracy. In fact, the BDF-scheme of order 2 is far less suitable for the solution than the implicit Euler-method.

The aforementioned behavior of the solution, in particular the appearance of sharp peaks, is not yet understood and has to be examined further.

7. Conclusions

The numerical calculations carried out in this paper strongly underline the necessity of a proper understanding of the stability properties of the time integration schemes when dealing with equations on a moving boundary. From the results of Section 4 we can conclude, that when working with ALE-methods implicit methods may not lead to unconditionally stable numerical schemes.

Furthermore, we have shown that for the linear advection-diffusion problem the modeling via the weak form (2.5) is more appropriate. This shows that the form of the finally discretized equation plays an important role in this context.

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